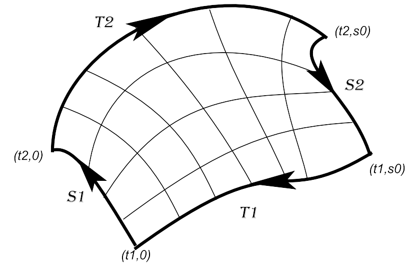

FIG. 1: Explanation to definition of bounds in integral (1.5).

I. CONSERVATION LAWS

In the previous lecture we derived the action of a free string:

$$S = \int_{\tau_1}^{\tau_2} d\tau \int_0^{\sigma_0} d\sigma \mathcal{L} \ , \quad \mathcal{L} = \sqrt{(\dot{x}\cdot x')^2 - x'^2\dot{x}^2} \ , \quad \dot{x}_\mu = \frac{dx_\mu}{d\tau} \ , \quad x'_\mu = \frac{dx_\mu}{d\sigma} \tag{1.1}$$

where $x_\mu(\sigma,\tau)$ is radius-vector of the string, τ is parameter characterizing the evolution of string, σ is an intrinsic coordinate of string ($\sigma = 0, \sigma_0$ are end points of the string). Also we derived the equation of motion for a free string:

$$\frac{\partial}{\partial\tau}\frac{\partial\mathcal{L}}{\partial\dot{x}_\mu} + \frac{\partial}{\partial\sigma}\frac{\partial\mathcal{L}}{\partial x'_\mu} = 0 \ , \tag{1.2}$$

$$\frac{\partial\mathcal{L}}{\partial x'_\mu}\Big|_{\sigma=0} = \frac{\partial\mathcal{L}}{\partial x'_\mu}\Big|_{\sigma=\sigma_0} = 0 \ . \tag{1.3}$$

Let us find the conservation laws corresponding to the Poincare group. The Poincare transformation has a form $x_\mu \to \omega_{\mu\nu}x^\nu + \epsilon_\mu$, where ϵ^μ is a constant vector, $\omega_{\mu\nu}$ is a matrix of Lorentz transformation (a matrix of rotation in Minkowski space). Varying the action (1.1), we find

$$\delta S = \int \Big(\frac{\partial\mathcal{L}}{\partial\dot{x}_\mu}\delta\dot{x}_\mu + \frac{\partial\mathcal{L}}{\partial\acute{x}_\mu}\delta x'_\mu\Big) d\tau d\sigma = 0. \tag{1.4}$$

Applying the Stock's formula

$$\int\int_{\text{area}} d\sigma d\tau \Big(\frac{\partial Q}{\partial\tau} - \frac{\partial P}{\partial\sigma}\Big) = \oint_{\text{bound}} \Big(P d\tau + Q d\sigma\Big)$$

with

$$Q = \frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}\delta x^\mu \quad \text{and} \quad P = -\frac{\partial\mathcal{L}}{\partial x'^\mu}\delta x^\mu \ ,$$

we obtain that the variation (1.4) is equivalent to:

$$-\int_{\text{area}} d\sigma d\tau \delta x_\mu \Big(\frac{\partial}{\partial\tau}\frac{\partial\mathcal{L}}{\partial\dot{x}_\mu} + \frac{\partial}{\partial\sigma}\frac{\partial\mathcal{L}}{\partial x'_\mu}\Big) + \oint_{T_1+S_1+T_2+S_2} \Big(\frac{\partial\mathcal{L}}{\partial\dot{x}^\mu}\delta x^\mu \, d\sigma - \frac{\partial\mathcal{L}}{\partial x'^\mu}\delta x^\mu \, d\tau\Big) = 0 \tag{1.5}$$

The S_1 and S_2 are bounds of world sheet with constant σ, and T_1,T_2 are bounds with constant τ. Applying the equations of motion (1.2) and (1.3) to (1.5) we find:

$$\int_{T_1+T_2} \delta x^\mu \Big(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\, d\sigma - \frac{\partial \mathcal{L}}{\partial x'^\mu}\, d\tau \Big) \;=\; 0 \;.$$

Because of T_1 and T_2 are lines of constant τ, $d\tau$ on that lines is equal to zero. The direction of T_1 is opposite to direction of T_2 (see. FIG.1), so we have:

$$\int_{T_1} \delta x_\mu \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\, d\sigma \;=\; \int_{T_2} \delta x_\mu \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\, d\sigma \;. \tag{1.6}$$

The momentum is a conserved quantity, which appears under transformations of a shift. Putting the $\delta x_\mu = \epsilon_\mu$, we find that quantity

$$P_\mu \;=\; \int_0^{\sigma_0} d\sigma \frac{\partial \mathcal{L}}{\partial \dot{x}_\mu}$$

is conserved, this is the full momentum of the string. Also it is useful to introduce a partial momentum

$$p_\mu(\tau,\sigma) \;=\; \frac{\partial \mathcal{L}}{\partial \dot{x}_\mu}\;, \tag{1.7}$$

which is not conserved.

Angular momentum corresponds to a quantity conserved under rotation: $\delta x_\mu = \omega_{\mu\nu} x_\nu$.

$$\int_{T_1} (x_\mu p_\nu - x_\nu p_\mu) d\sigma \;=\; \int_{T_2} (x_\mu p_\nu - x_\nu p_\mu) d\sigma \;,$$

$$J_{\mu\nu} \;=\; \int_0^{\sigma_0} d\sigma (x_\mu p_\nu - x_\nu p_\mu).$$

Note, that one may put T_1 and T_2 as arbitrary lines $T(\tau,\sigma)$, which cross the world sheet of a sting from border S_1 to border S_2. And the conservation law would be still valid in the form:

$$P_\mu \;=\; \int_T \Big(\frac{\partial \mathcal{L}}{\partial \dot{x}_\mu} d\sigma - \frac{\partial \mathcal{L}}{\partial x'_\mu} d\tau \Big)$$

Exercise. Derive the conservation law for an angular momentum on arbitrary line T.

II. HAMILTON FORMALISM

In a lot of applications of the string theory one needs to construct the Hamilton formalism for a string. But it is not so obvious, because a string is a system with constrains (primary constrains). This constrains are connected with the invariance of the string action under the transformations ($\sigma \to \sigma'(\sigma,\tau)$, $\tau \to \tau'(\sigma,\tau)$).

First, let us consider more simple (but very important) system with constrains. It is a free relativistic particle. As it was discussed in previous lecture the action for a free particle is

$$S \;=\; -\, mc \int d\tau \sqrt{\dot{x}^2} \;. \tag{2.1}$$

Its momentum is

$$p_\mu \;=\; \frac{\partial \mathcal{L}}{\partial \dot{x}_\mu} \;=\; -\, mc \frac{\dot{x}_\mu}{\sqrt{\dot{x}^2}} \; . \tag{2.2}$$

The turning from the Lagrange formalism to the Hamilton formalism is connected with turning from a pair of variables $(x_\mu, \dot{x}_\mu)$ to (x_μ, p_μ). But the equation (2.2) is unsolvable with respect to $\dot{x}_\mu$. One can easy see that Jacobian of transformation is singular (i.e.$\det(\frac{D(p_\mu)}{D(\dot{x}_\mu)}) \;=\; 0$). If we try to build the Hamiltonian as usual (against all the odds), we immediately find that it is zero:

$$H_0 \;=\; p_\mu \dot{x}_\mu - \mathcal{L} \;=\; 0$$

For the string action we have same situation. Why does it happen? The deep reason for this is the invariance of action (2.1) under the reparameterization $\tau \to \tau'(\tau)$. It implies appearance of constrains which has a form $f(p, x) = 0$. Squaring both side of (2.2), we obtain $p^2 \;=\; m^2 c^2$. A free particle moves in such way that its momentum is always "on mass shell".

There are two methods to avoid this problem. The first method (called non-covariant) consists in the reducing of degrees of freedom number with respect of constrains, and after that construct the Hamiltonian as usual. But this method brakes explicitly initial symmetries of model (for example Lorentz symmetry for free particle). The second method lies in Lagrange multiples. We add to the Lagrangian the term $\sum_n \lambda_n(p, x) f_n(p, x)$, where f_n are constrains, λ_n are arbitrary function. After that, we construct the Hamiltonian as usual, but the equations of motion now contain additional unknowns (λ's). This effectively reduces the number of equations, but still preserve all symmetries.

Let us realize this program. At first, we derive equations of motion for the string in the Hamiltonian formalism. Varying the Hamiltonian, we find

$$\delta H \;=\; \frac{\partial H}{\partial p_\mu}\delta p_\mu + \frac{\partial H}{\partial x'_\mu}\delta x'_\mu \;=\; \dot{x}_\mu \delta p_\mu - \frac{\partial \mathcal{L}}{\partial x'_\mu}\delta x'_\mu + (p_\mu \delta \dot{x}_\mu - \frac{\partial \mathcal{L}}{\partial \dot{x}_\mu}\delta \dot{x}_\mu) \; .$$

The term in the brackets turns to zero due to definition (1.7). That gives two equations:

$$\dot{x}_\mu \;=\; \frac{\partial H}{\partial p_\mu} \; , \quad \dot{p}_\mu \;=\; \frac{\partial}{\partial \sigma}\frac{\partial H}{\partial x'_\mu} \; . \tag{2.3}$$

Also we have an equivalent of (1.3):

$$\frac{\partial H}{\partial x'_\mu}\Big|_{\sigma=0} \;=\; \frac{\partial H}{\partial x'_\mu}\Big|_{\sigma=\sigma_0} \;=\; 0$$

This is a necessary equations.

At second, let us find explicit form of constrains. According to the (1.7), we have

$$p_\mu \;=\; A \frac{(\dot{x} \cdot x')x'_\mu - x'^2 \dot{x}_\mu}{\sqrt{(\dot{x} \cdot x')^2 - \dot{x}^2 x'^2}} \; .$$

It is easy to see that

$$(p \cdot p) \;=\; -\, A^2 x'^2 \; , \quad (p \cdot x') \;=\; 0 \; . \tag{2.4}$$

We rewrite it in the following way:

$$\phi_\pm(p,x') \;=\; (p_\mu \pm A x'_\mu)^2 \;= 0 \;.$$

Let us note that $\phi_\pm$ are equal to zero in a weak sense. It means that one have to make all calculation with them as usual and put them to zero only in the very end.

As it was discussed we construct the Hamiltonian in the form:

$$H \;=\; H_0 - \mathcal{L} + \lambda_+(p,x')\phi_+(p,x') + \lambda(p,x')_-\phi_-(p,x')$$

By direct calculations one can find that H_0 is zero. Substituting H into equations of motion (2.3), we find:

$$\dot{x}_\mu \;=\; 2(\lambda_+ + \lambda_-)p_\mu + 2A(\lambda_+ - \lambda_-)x'_\mu \;,$$

$$\dot{p}_\mu \;=\; 2A\frac{d}{d\sigma}\Big[A(\lambda_+ + \lambda_-)x'_\mu + (\lambda_+ - \lambda_-)p_\mu\Big] \;.$$

The next task is to solve this equation with respect of $\lambda_\pm$. Multiplying both sides by x'_μ we find that

$$(\dot{x}\cdot x') \;=\; 2Ax'^2(\lambda_+ - \lambda_-) \;\Rightarrow\; \lambda_+ - \lambda_- \;=\; \frac{(\dot{x}\cdot x')}{2Ax'^2} \;.$$

Squaring both sides and using that $(p\cdot x') = 0$ we obtain

$$\dot{x}^2 \;=\; 4(\lambda_+ + \lambda_-)^2p^2 + 4A^2(\lambda_+ - \lambda_-)^2x'^2 \;\Rightarrow\; \lambda_+ + \lambda_- \;=\; \frac{\sqrt{(\dot{x}\cdot x')^2 - x'^2\dot{x}^2}}{2Ax'^2} \;.$$

In the orthonormal parameterization (see previous lecture) the Lagrange multiplies are fixed:

$$\lambda_+ \;=\; \lambda_- \;=\; -\frac{1}{4A} \;.$$

Exercise. Using the obtained Lagrange multiplies, show explicitly that Hamilton equations are equivalent to the Lagrange equation.

III. SOLUTION OF THE EQUATION OF MOTION

Let us discuss the solution of equation of motion. In the ortonormal system of coordinates ($(\dot{x}_\mu \pm x'_\mu)^2 \;=\; 0$) equations (1.2) and (1.3) have a form

$$\ddot{x}_\mu - x''_\mu \;=\; 0 \;,\quad x'_\mu(\tau,0) \;=\; x'_\mu(\tau,\sigma_0) \;=\; 0 \;. \tag{3.1}$$

The general solution of the first equation is

$$x_\mu(\sigma,\tau) \;=\; r_\mu\tau + f_\mu(\tau+\sigma) + f_\mu(\tau-\sigma) \;. \tag{3.2}$$

Here r_μ in principal can be put to zero, because such term already exists in last two terms ($f_\mu(z) = r_\mu z$). But we will keep it separately, the special meaning of r_μ will describe latter.

Substituting the general solution into the second pair of equations (3.1), we find that f'_μ (derivative of f_μ) should be periodical function:

$$f'_\mu(\tau) \;=\; f'_\mu(\tau + 2\sigma_0) \;.$$

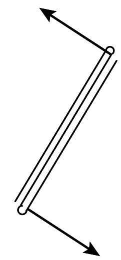

FIG. 2: A folded rotating string, $N = 3$.

The condition of a ortonormal parameterization gives us additional constrain:

$$(r_\mu + 2f'_\mu(z))^2 \;=\; 0\;.$$

With help of knowledge of string initial state ($x_\mu(0,\sigma)$ and $\dot{x}_\mu(0,\sigma)$) one can fix the function f_μ fully:

$$f_\mu(z) \;=\; \frac{1}{2}\Big[x_\mu(0,|z|) + \mathrm{sign}(z)\int_0^{|z|}\dot{x}(0,\sigma)d\sigma - r_\mu z\Big]\;, \tag{3.3}$$

where the vector r_μ is related to the full momentum of the sting as

$$r_\mu \;=\; \frac{1}{\sigma_0}\int_0^{\sigma_0}\dot{x}_\mu(0,\sigma)d\sigma \;=\; \frac{1}{A\sigma_0}P_\mu\;.$$

Note, that in the laboratory frame (when $x_0 \;=\; c\tau$) we have $f_0 = 0$ and $r_\mu \;=\; (c,0,0,0)$, so we can define the mass of the string

$$M \equiv A\sigma_0\;.$$

Let us consider one particular case. We chose the initial state for a string as

$$x_\mu(0,\sigma) \;=\; \frac{W_\mu}{\omega_N}\cos(\omega_n\sigma)\;,\quad \dot{x}(0,\sigma) \;=\; r_\mu + V_\mu\cos(\omega_N\sigma)\;,$$

where

$$\omega_N \;=\; \frac{N\pi}{\sigma_0}\;,\quad N \;=\; 1,\;2,\;3..,$$

W_μ and V_μ are some constant vectors. One can easy understand that such initial condition describes a string N times folded.

In order to keep our bound condition self consistent, we put $(x'\dot{x}) = 0$ and $\dot{x}^2 + x'^2 = 0$ that gives us:

$$(rV) \;=\; (rW) \;=\; (VW) \;= 0\;,\quad r^2 \;=\; -V^2 \;=\; -W^2\;.$$

Let us also turn to the center of mass frame, where the sting as a whole rests

$$\vec{P} \;=\; 0\;,\quad P_0 \;=\; Mc\;.$$

In this frame we have $\vec{r}=0$ and $V_0 = W_0 = (\vec{V}\vec{W}) = 0$. Also we introduce the "laboratory" parameterization, $x_0 = c\tau$. As it was discussed in this frame $f_0 = 0$ and $r_\mu = (1,0,0,0)$. Finally, collecting all together and puting into (3.3) and (3.2) we obtain

$$\vec{f}(\sigma) = \frac{1}{2\omega_N}\left[\vec{W}\cos(\omega_N\sigma) + \vec{V}\sin(\omega_N\sigma)\right] ,$$

$$x_0(\tau,\sigma) = c\tau \ , \quad \vec{x}(\tau,\sigma) = \frac{\cos(\omega_N\sigma)}{\omega_N}\Big(\vec{W}\cos(\omega_N\tau) + \vec{V}\sin(\omega_N\tau)\Big) \ . \qquad (3.4)$$

Using this solution, one can easy calculate that the matrix of angular momentum for the string is

$$J_{\mu\nu} = \begin{pmatrix} 0 & 0 \\ 0 & [\vec{W}\times\vec{V}]\frac{A\sigma_0^2}{2\pi N} \end{pmatrix} .$$

This means that our string rotates in the plane $(\vec{V},\vec{W})$. The modules of angular momentum is

$$J = \frac{A\sigma_0^2}{2\pi N} = \frac{M^2}{2\pi c^2 AN} .$$

and it is proportional to mass squared! One can show that for any boundary conditions $J \lesssim M^2$. This situation is typical for particle physics. And it was one of strong point to offer a string as a model of elementary particle.

Exercise. Show that for solution (3.4) the length of a string do not depend on time.