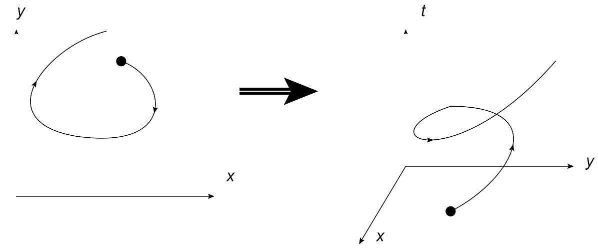

FIG. 1: The trajectory of a particle and its world line.

I. RELATIVISTIC ACTION FOR A FREE POINT PARTICLE

The Principle of Equivalence says that all physic laws should be identical in any frame. In classical (non-relativistic) mechanic any change of frame is generated by the Gallilean transformations $\vec{r}' = \vec{r} - \vec{v}t$, $t' = t$. The Gallilean transformations keep invariant distance between two points $\Delta\vec{r}^2 = \text{inv} = \Delta r_1^2 + \Delta r_2^2 + \Delta r_3^2$ and changes velocities $\dot{\vec{r}}' = \dot{\vec{r}} - \vec{v}$. The special theory of relativity postulates that the speed of light is same in any frame. One has to use Lorentz transformations instead of Gallilean to satisfy this requirement.

The Lorentz transformations keep invariant the interval:

$$c^2\Delta t^2 - \Delta x_1^2 - \Delta x_2^2 - \Delta x_3^2 = \text{inv}$$

Usually people write instead of ct (time) just x_0, a zero component of vector in 4D space-time (4-vector). The scalar product of two 4-vectors is defined as

$$(x \cdot y) = x_0 y_0 - x_1 y_1 - x_2 y_2 - x_3 y_3 \ . \tag{1.1}$$

Four dimensional space with scalar production defined as (1.1) is called Minkowski space. There is a number of relativistic effects followed from the Lorentz transformation. For example, the time dilatation and the longitudinal size contraction for fast moves bodies [1].

Let us describe the motion of the particle with help of the radius vector $x_\mu(\tau)$ in the space-time. During the motion from the initial ($x_\mu(\tau_1)$) to the final state ($x_\mu(\tau_2)$) it draws some curve in the Minkowski space. This curve is called world line. The length of the world line is Lorentz invariant quantity.

We define the action as a length of this line.

$$S = A\int dl = A\int\sqrt{\frac{dx_0(\tau)}{d\tau}\frac{dx_0(\tau)}{d\tau} - \frac{d\vec{x}(\tau)}{d\tau}\frac{d\vec{x}(\tau)}{d\tau}}d\tau \ .$$

Obviously the action does not depend on a parameterization of the world line.

$$S = A\int\sqrt{\frac{\partial x_\mu}{\partial\tau'}\frac{\partial x_\mu}{\partial\tau'}}d\tau' = A\int\sqrt{\frac{\partial x_\mu}{\partial\tau}\frac{\partial x_\mu}{\partial\tau}\left(\frac{\partial\tau}{\partial\tau'}\right)^2}\frac{\partial\tau'}{\partial\tau}d\tau = A\int\sqrt{\frac{\partial x_\mu}{\partial\tau}\frac{\partial x_\mu}{\partial\tau}}d\tau$$

In the laboratory frame, when $x_0 = c\tau$ (the time is measured by observer's clock), we have:

$$S = Ac\int\sqrt{1 - \frac{1}{c^2}\frac{d\vec{x}(t)}{dt}\frac{d\vec{x}(t)}{dt}}dt = Ac\int\sqrt{1 - \frac{v^2}{c^2}}dt \tag{1.2}$$

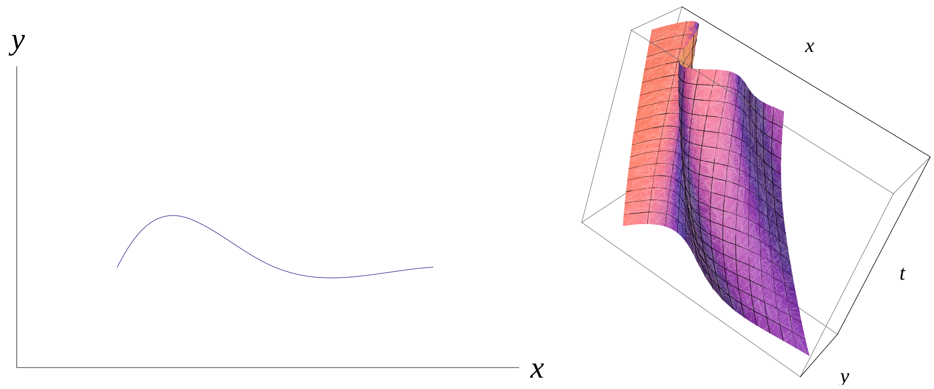

FIG. 2: The moving string sweeps a surface in Minkowski space.

Assuming $v \ll c$ (non relativistic limit) and expanding (1.2), we get

$$S \;=\; A\int dt(c - \frac{v^2}{2c} + \mathcal{O}(\frac{v^4}{c^4})) \qquad (1.3)$$

This expression we can compare with the classical Lagrangian for a free particle ($\mathcal{L}_{\text{non-R}} = \frac{mv^2}{2}$). We find that $A = -mc$.

II. ACTION FOR A CLASSICAL STRING

A string is a one-dimension object in the Minkowski space. Every point of string can be parameterized by the radius vector:

$$x_\mu(\sigma)\ ,\quad \mu = 0, 1, 2, 3\ .$$

The parameter σ is an intrinsic coordinate of the string, the ends of the string correspond to $\sigma = 0, \sigma_0$. We also know boundary conditions: the initial state of string $x_\mu(\tau_1, \sigma)$ and final state $x_\mu(\tau_2, \sigma)$, where τ is time of evolution of the string.

During the evolution from the initial to the final state the string sweeps in 4D space-time the "world" surface. Element of this surface is

$$d^2s \;=\; \sqrt{(\dot{x}\cdot\acute{x})^2 - (\dot{x}\cdot\dot{x})(x'\cdot x')}d\sigma d\tau\ ,$$

where

$$\dot{x}_\mu \;=\; \frac{\partial x_\mu(\sigma,\tau)}{\partial\tau}\ ,\quad x'_\mu \;=\; \frac{\partial x_\mu(\sigma,\tau)}{\partial\sigma}$$

and the scalar product is defined in (1.1).

In the full analogy of a point particle action we define the action for a string as an area of its world sheet:

$$S \;=\; -A\int d^2s \;=\; -A\int_{\tau_1}^{\tau_2}\int_0^{\sigma_0}\sqrt{(\dot{x}\cdot x')^2 - (\dot{x}\cdot\dot{x})(x'\cdot x')}d\sigma d\tau\ . \qquad (2.1)$$

The A has a dimension m^2/t and proportional to the string tension. It will be discussed in the next lecture.

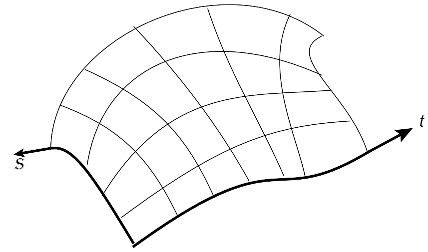

FIG. 3: A parameterization of a surface.

A very important note is that the action of the string does not depend on parameterization (τ, σ). We can choose any parameterization $(\tau'(\tau, \sigma), \sigma'(\tau, sigma))$. The only constrain on the reparameterization is that the bounds of surface should keeps changless: $\sigma'(\tau, 0) = 0$, $\sigma'(\tau, \sigma_0) = \sigma_0$. Physical quantities do not depend on this freedom, and it is better to fix parameterization at very beginning. $\boxed{Exercise.\ Show\ explicitly\ that\ area\ of\ surface\ does\ not\ depend\ on\ parameterization?}$

One of the most simple parameterization is the orthonormal parameterization, then the grid lines for σ and τ orthogonal to each other at every point of surface. We put

$$(\dot{x} \cdot x') = 0 , \quad (\dot{x} \cdot \dot{x}) = -(x' \cdot x') .$$

Applying this condition to the (2.1), we find that

$$S = -A \int d\tau d\sigma (\dot{x} \cdot \dot{x})$$

In the laboratory frame $(x_0 = \tau)$ we have

$$S = -Ac^2 \int d\tau d\sigma \Big(1 - \frac{\vec{v}^2(\tau, \sigma)}{c^2}\Big).$$

An element of the proper length for the string at the time τ is

$$dL_0(\tau) = \sqrt{(x' \cdot x')} d\sigma = \sqrt{1 - \frac{v^2}{c^2}} d\sigma$$

Due to the Lorentz length contraction an element of the string length in the laboratory frame is shorter

$$dL = \sqrt{1 - \frac{v^2}{c^2}} dL_0 = \Big(1 - \frac{\vec{v}^2}{c^2}\Big) d\sigma$$

Combining this formulae together we obtain the total length of the string in the laboratory frame:

$$L(\tau) = \int dL = \int_0^{\sigma_0} \Big(1 - \frac{\vec{v}^2}{c^2}\Big) d\sigma$$

The first observation is that the length of a free string is not conserve during the movement. The action can be rewritten as

$$S = -Ac^2 \int d\tau L(\tau)$$

Due to the principal of least action one can say that a string at every moment of its evolution "wants" to have as minimal length as possible. So, in principal, it is better to imagine that a string is a spring.

III. EQUATIONS OF MOTION FOR A FREE STRING

The principle of least action says:

$$\delta S \;=\; \int \delta\mathcal{L} d\sigma d\tau \;=\; 0 \;=\; \int \Big(\frac{\partial\mathcal{L}}{\partial\dot{x}_\mu}\delta\dot{x}_\mu \;+\; \frac{\partial\mathcal{L}}{\partial\acute{x}_\mu}\delta x'_\mu\Big) d\tau d\sigma \;.$$

Integrating by parts we obtain:

$$0 = \;\int \delta x_\mu \Big(- \frac{\partial}{\partial\tau}\frac{\partial\mathcal{L}}{\partial\dot{x}_\mu} - \frac{\partial}{\partial\sigma}\frac{\partial\mathcal{L}}{\partial x'_\mu}\Big) + \int d\sigma\Big(\delta x_\mu \frac{\partial\mathcal{L}}{\partial\dot{x}_\mu}\Big)\Big|_{\tau=\tau_1}^{\tau_2} + \int d\tau\Big(\delta x_\mu \frac{\partial\mathcal{L}}{\partial x'_\mu}\Big)\Big|_{\sigma=0}^{\sigma_0}$$

The $\delta x_\mu(\tau,\sigma)$ is an arbitrary function, with $\delta x_\mu(\tau_1,\sigma) = \delta x_\mu(\tau_2,\sigma) = 0$(fixed initial and final states). But the end-points of a string are free, it means that $\delta x_\mu(\tau,0)$ and $\delta x_\mu(\tau,\sigma_0)$ are also an arbitrary functions. That gives us three equations of motions:

$$\frac{\partial}{\partial\tau}\frac{\partial\mathcal{L}}{\partial\dot{x}_\mu} + \frac{\partial}{\partial\sigma}\frac{\partial\mathcal{L}}{\partial x'_\mu} \;=\; 0 \;, \tag{3.1}$$

$$\frac{\partial\mathcal{L}}{\partial x'_\mu}\Big|_{\sigma=0} \;=\; \frac{\partial\mathcal{L}}{\partial x'_\mu}\Big|_{\sigma=\sigma_0} \;=\; 0 \;. \tag{3.2}$$

Let us investigate the last two equations of motion. For this let's calculate the following quantity $\left(\frac{\partial\mathcal{L}}{\partial x'_\mu}\cdot\frac{\partial\mathcal{L}}{\partial x'_\mu}\right)$ for the end-points of string. Using the explicit form of the Lagrangian density (2.1) , we find:

$$\Big(\frac{\partial\mathcal{L}}{\partial x'_\mu}\cdot\frac{\partial\mathcal{L}}{\partial x'_\mu}\Big) \;=\; A^2(\dot{x}\cdot\dot{x})$$

According to the equations of motion (3.2) at the end points of the string , this quantity is equal to zero:

$$\Big(\frac{\partial\mathcal{L}}{\partial x'_\mu}\cdot\frac{\partial\mathcal{L}}{\partial x'_\mu}\Big)\Big|_{\sigma=0,\sigma_0} \;=\; (\dot{x}\cdot\dot{x})\Big|_{\sigma=0,\sigma_0} \;=\; 0$$

What does it mean? It is easy to understand in laboratory frame:

$$1 - \frac{v^2}{c^2}\Big|_{\sigma=0,\sigma_0} \;=\; 0, \;\;\Rightarrow |v|\Big|_{\sigma=0,\sigma_0} \;=\; c$$

The end points of the string are moves with the speed of light.

Exercise. Derive the explicit form of the first equation of motion(3.1).

[1] L.D.Landau, E.M.Lifschitz *Lehrbuch der theoretischen Physik* (Band 2).