Lection 7.

I. EQUATION FOR A SWING: NON-PERTURBED CASE.

We just remind you some results of the previous lecture. To demonstrate how the parametric resonance works, we consider the Mathieu's equation for one-dimensional oscillation system:

$$\ddot{x} + \omega_0^2\big(1 + h\cos(\gamma t)\big)x \; = \; 0, \tag{1.1}$$

where frequency ω oscillates around of ω_0 with a small amplitude. According to methods of perturbation theory, we started our consideration from equation with $h = 0$:

$$\ddot{x} + \omega_0^2 x \; = \; 0, \tag{1.2}$$

and obtained the monodromy matrix $\hat{A}$ for it:

$$\hat{A}\left(\frac{2\pi}{\gamma}\right) = \begin{pmatrix} \cos(\frac{2\pi\omega_0}{\gamma}) & \frac{1}{\omega_0}\sin(\frac{2\pi\omega_0}{\gamma}) \\ -\omega_0\sin(\frac{2\pi\omega_0}{\gamma}) & \cos(\frac{2\pi\omega_0}{\gamma}) \end{pmatrix}. \tag{1.3}$$

This matrix corresponds to an evolution of the system (1.2) for time $t = \frac{2\pi}{\gamma}$. Analyzing the matrix, we obtained:

- The system is always stable, because of $|\mathrm{Tr}(\hat{A})| \leq 2$;
- From the condition $|\mathrm{Tr}(\hat{A})| = 2$ we have got, that regions of instability (where the parametrical resonance works) lie around points $\gamma = \frac{2\omega_0}{k}$ or, in other words, $\omega_0 = \frac{\gamma}{2}, \gamma, \frac{3\gamma}{2}....$

II. EQUATION FOR A SWING: FIRST CORRECTION.

The derivation for the first correction in h for eq.(1.1) can be found in L.D.Landau and E.M.Lifschitz (V.1) §27. Here we present the perturbation theory for a bit different case. But, as for eq.(1.1), we can expect same effects when the perturbation is switched on. Namely that the frequency should be inside of intervals:

$$\gamma = \frac{2\omega_0}{k} \pm f(h).$$

Let us replace the cosine in (1.1) by the step-function (see Fig.1):

$$\begin{aligned} \ddot{x} \; + \; \omega^2(t)x = 0, \\ \omega(t) \;\; = \begin{cases} \omega_0 + h, & 0 < t < \pi \\ \omega_0 - h, & \pi < t < 2\pi. \end{cases} \end{aligned} \tag{2.1}$$

In this case to get the mapping for a period we have to make mapping twice for a half-period:

$$\vec{z}(T) = \hat{A}(T)\vec{z}(0) = \hat{A}_2\hat{A}_1\vec{z}(0). \tag{2.2}$$

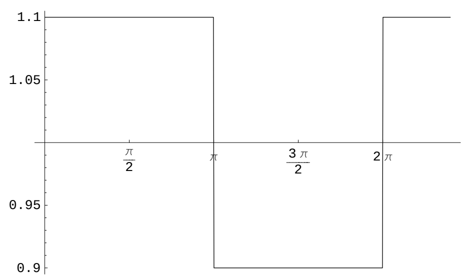

FIG. 1: The evolution of frequency with time.

Both monodromy matrices are mappings for a half-period for the system with the constant frequency $\omega_1 = \omega_0 + h$ and $\omega_2 = \omega_0 - h$ and have same form as a monodromy matrix for constant frequency in non-perturbed case (1.3). So, we get $\hat{A}$ as a product:

$$\hat{A}(2\pi) = \begin{pmatrix} \cos(\pi\omega_2) & \frac{1}{\omega_2}\sin(\pi\omega_2) \\ -\omega_2\sin(\pi\omega_2) & \cos(\pi\omega_2) \end{pmatrix} \begin{pmatrix} \cos(\pi\omega_1) & \frac{1}{\omega_1}\sin(\pi\omega_1) \\ -\omega_1\sin(\pi\omega_1) & \cos(\pi\omega_1) \end{pmatrix}. \qquad (2.3)$$

Let us find the bounds of the stability for eq. (2.1) and regions, where the system is unstable because of the parametric resonance. We use condition $\mathrm{Tr}(\hat{A}) = \pm 2$:

$$\left[2\cos(\pi\omega_1)\cos(\pi\omega_2) - \sin(\pi\omega_1)\sin(\pi\omega_1)\left(\frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1}\right)\right] = \pm 2. \qquad (2.4)$$

Then let's expand in the Taylor's series up to h^2 the expression in the brackets:

$$\left(\frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1}\right) = 2 + 2h^2/\omega_0^2 + O(h^4) + ... \simeq 2(1+\Delta), \qquad \Delta = \frac{2h^2}{\omega_0^2},$$

and using standard trigonometrical relations we get

$$\cos(2\pi\omega_0)(2+\Delta) = \pm 2 + \Delta\cos(2\pi h). \qquad (2.5)$$

- Let us choose, in the beginning, the sign "+" and rewrite above expression as

$$\cos(2\pi\omega_0) = 1 - \frac{\Delta}{(2+\Delta)}(1 - \cos(2\pi h)),$$
$$\cos(2\pi\omega_0) = 1 - \frac{(\pi\omega_0)^2\Delta^2}{(2+\Delta)}, \qquad (2.6)$$

where smallness of h was used. Now let us compare this with a famous trigonometrical formula, when $\alpha << 1$:

$$\cos(2\alpha) = 1 - 2\sin^2(\alpha) = 1 - 2\alpha^2 \qquad (2.7)$$

and we get

$$\alpha = \frac{\pi\omega_0\Delta}{\sqrt{2(2+\Delta)}} \simeq \frac{\pi\omega_0\Delta}{2}$$

Then we can find from (2.6) and (2.7)ω_0:

$$
\begin{aligned}
2\pi\omega_0 &= 2\alpha \pm 2\pi k = \pi\omega_0\Delta \pm 2\pi k, \quad \Delta = \frac{2h^2}{\omega_0^2}, \\
\omega_0^2 &= h^2 \pm \omega_0 k, \\
\omega_0 &= k \pm \frac{h^2}{k}, \qquad\qquad k = 1, 2, 3... \qquad (2.8)
\end{aligned}
$$

- For the case with "$-$" in (2.5):

$$
\omega_0 = k + \frac{1}{2} \pm \frac{h}{\pi\left(k+\frac{1}{2}\right)}, \qquad k = 0, 1, 2... \qquad (2.9)
$$

Problem : Obtain this answer yourself!

Compare eq. (2.8) with eq. (2.9): the first is quadratic in h, but second is linear. It is means that the bounds of unstable areas of parameters have different form for integer and half-integer ω_0 (here $\gamma = 1$ for simplicity). Moreover, the regions of parametrical resonance decrease (become narrow) with k:

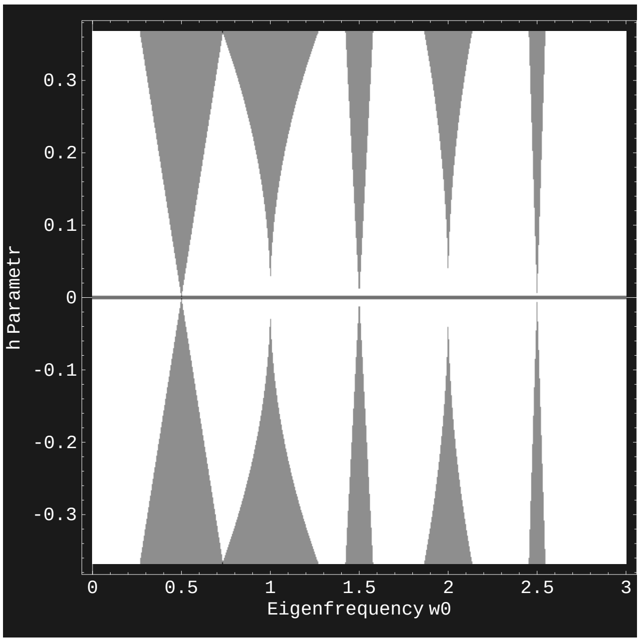

FIG. 2: The resonance regions.

In the case of the Mathieu's equation the bounds of stable will be more curved and the parametrical resonance will be observed only on $2\omega_0$ and ω_0 frequencies.