Lection 6.

I. PARAMETRIC RESONANCE.

We remind you: to investigate the parametric resonance, we consider a equation for one-dimensional oscillation system, when its frequency is changing with time periodically:

$$\ddot{x} + \omega^2(t)x \;=\; 0, \qquad\qquad \omega^2(t) = \omega_0^2\big(1 + h\cos(\gamma t)\big), \tag{1.1}$$

where $h << 1$, i.e. frequency ω oscillates around of ω_0 with a very small amplitude. As differential equation of the second order eq.(1.1) has two linearly-independent particular solutions: $x_1(t)$ and $x_2(t)$. In this way we construct a general deviation and a momentum:

$$\begin{aligned} x(t) &= c_1x_1(t) + c_2x_2(t) \\ \dot{x}(t) &= c_1\dot{x}_1(t) + c_2\dot{x}_2(t). \end{aligned} \tag{1.2}$$

It is useful for below to rewrite this system of equations in the matrix form:

$$\begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} = \begin{pmatrix} x_1(t) & x_2(t) \\ \dot{x}_1(t) & \dot{x}_2(t) \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \hat{W}(t)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \tag{1.3}$$

where we denote the 2×2 matrix as $\hat{W}(t)$, because the determinant of this matrix is named Wronskian (see the last lecture): $\det(\hat{W}(t)) = W(t)$.
While the frequency $\omega(t)$ is periodical, $x_{1,2}(t+T)$ are also solutions of eq.(1.1) and can be expressed through a linear combination of $x_{1,2}(t)$:

$$\begin{aligned} x_1(t+T) &= \lambda_1x_1(t) + \nu_1x_2(t) \\ x_2(t+T) &= \lambda_2x_1(t) + \nu_2x_2(t). \end{aligned} \tag{1.4}$$

Then, substituting it in eq.(1.3), we get

$$\begin{aligned} \begin{pmatrix} x(t+T) \\ \dot{x}(t+T) \end{pmatrix} &= \begin{pmatrix} \lambda_1x_1(t) + \nu_1x_2(t) & \lambda_2x_1(t) + \nu_2x_2 \\ \lambda_1\dot{x}_1(t) + \nu_1\dot{x}_2 & \lambda_2\dot{x}_1(t) + \nu_2\dot{x}_2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \\ &= \begin{pmatrix} x_1(t) & x_2(t) \\ \dot{x}_1(t) & \dot{x}_2(t) \end{pmatrix}\begin{pmatrix} \lambda_1(t) & \lambda_2(t) \\ \nu_1(t) & \nu_2(t) \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}. \end{aligned} \tag{1.5}$$

Or we can write it in more short form:

$$\begin{aligned} \vec{z}(t) &= \hat{W}(t)\vec{c} \\ \vec{z}(t+T) &= \hat{W}(t)\hat{B}(T)\vec{c}, \end{aligned} \tag{1.6}$$

where we have denoted the matrix of the coefficients as $\hat{B}(T)$. We also have introduced the vector $\vec{z}(t)$, which lives in the Phase Space, the two-dimensional space of the coordinate and momentum. Now we show, that mechanical systems preserve a phase volume(a phase area in our case). From this fact follows a very important property of matrix $\hat{B}(T)$.

II. LIOUVILLE'S THEOREM ABOUT CONSERVATION OF A PHASE VOLUME.

Let us consider system with the following equation of motion:

$$\ddot{x} + f(x,t) = 0, \tag{2.1}$$

in our case (1.1) $f(x,t) = \omega^2(t)x$. It is easy to see, that this equation can be rewritten as a vectorial equation in the phase space:

$$\begin{aligned} \dot{\vec{z}}(t) &= \vec{F}(\vec{z},t), \qquad (2.2) \\ \vec{z}(t) = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}, &\qquad \vec{F}(\vec{z},t) = \begin{pmatrix} z_2 \\ -f(z_1,t) \end{pmatrix}. \end{aligned}$$

According to Liouville's theorem: an area of the phase space is conserved $S(0) = S(t)$ (see Fig.1):

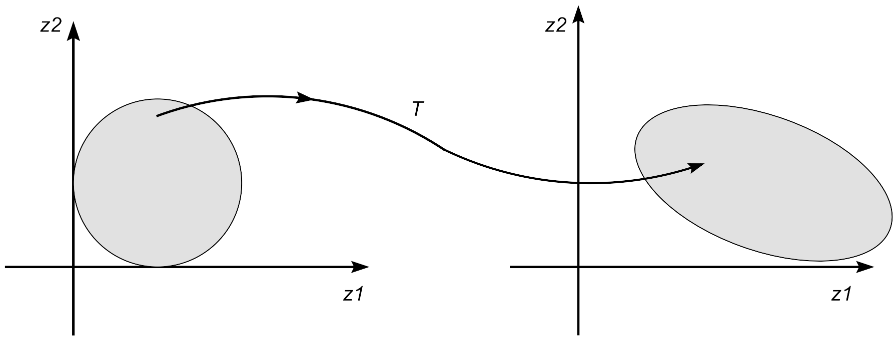

FIG. 1: The evolution of initial conditions with time.

$$\begin{aligned} S(0) &= \int dz_1(0)dz_2(0) = \int d\vec{z}(0) \\ S(t) &= \int dz_1(t)dz_2(t) = \int d\vec{z}(t) = \int d\vec{z}(0) \det\left(\frac{\partial z_i(t)}{\partial z_j(0)}\right), \end{aligned} \tag{2.3}$$

where after changing of variables the determinant appears - Jacobian. To calculate the Jacobian, let us expand $\vec{z}(t)$ around zero:

$$\vec{z}(t) = \vec{z}(0) + t\dot{\vec{z}}(0) + ... = \vec{z}(0) + t\vec{F}(\vec{z},0) + ... \tag{2.4}$$

Then

$$\frac{\partial z_i(t)}{\partial z_j(0)} = \delta_{ij} + t\frac{\partial F_i(t)}{\partial z_j(0)} = \mathrm{I} + t\hat{F}, \tag{2.5}$$

$$\det(\mathrm{I} + t\hat{F})\big|_{t\to 0} = e^{\mathrm{Tr}\log(\mathrm{I}+t\hat{F})} \approx e^{t\mathrm{Tr}\hat{F}} = 1 + t\mathrm{Tr}\hat{F}, \tag{2.6}$$

and from the definition of the matrix $\hat{F}$ (2.5):

$$\mathrm{Tr}\hat{F} = \mathrm{Tr}\left(\frac{\partial F_i(t)}{\partial z_j(0)}\right) = \sum_{i=1}^{2} \frac{\partial F_i(t)}{\partial z_i(0)} = div\vec{F} = \frac{\partial z_2}{\partial z_1} + \frac{\partial f(z_1,0)}{\partial z_2} = 0. \tag{2.7}$$

So, we have obtained the Jacobian of transformation is 1 and phase areas are conserved under evolution of the system in time. Moreover, now we know more about matrix $\hat{B}$ (1.6): its determinant is Jacobian of the transformation for a period and

$$\det(\hat{B}(T)) = 1. \tag{2.8}$$

III. MAPPING FOR A PERIOD.

In the first part the matrix $\hat{B}$ was introduced as (1.6). This matrix determine how the phase area will be transformed for a period T. And it doesn't matter from what time we are starting, let initial time be $t = 0$:

$$\begin{cases} \vec{z}(0) = \hat{W}(0)\vec{c} \\ \vec{z}(T) = \hat{W}(0)\hat{B}(T)\vec{c} \end{cases} \tag{3.1}$$

In literature one use a little bit different definition of the mapping for a period:

$$\vec{z}(T) = \hat{A}(T)\vec{z}(0) = \hat{A}(T)\hat{W}(0)\vec{c}, \tag{3.2}$$

but these mappings (other name is monodromy matrix) are related by simple transformation and all important properties are same:

$$\begin{aligned} \hat{A}(T) &= \hat{W}(0)\hat{B}(T)\hat{W}^{-1}(0) \\ \det(\hat{A}) &= \det(\hat{B}) = 1. \end{aligned} \tag{3.3}$$

Now we can predict a behaviour of the system from investigation of the monodromy matrix. The regimes are related to its eigenvalues. Let us solve characteristic equation for matrix $\hat{A}$:

$$\det(\hat{A} - \mu \mathrm{I}) = \mu^2 - \mathrm{Tr}(\hat{A}) + 1 = 0. \tag{3.4}$$

$$\mu_{1,2} = \frac{\mathrm{Tr}(\hat{A})}{2} \pm \sqrt{\left(\frac{\mathrm{Tr}(\hat{A})}{2}\right)^2 - 1}.$$

(Note: eigenvalues for $\hat{A}$ and $\hat{B}$ are identical)

- $|\mathrm{Tr}(\hat{A})| < 2$ corresponds to periodic solution with imaginary eigenvalues: $\mu_{1,2} = e^{\pm i\alpha}$. The mapping is a rotation on α angle and is stable.
- $|\mathrm{Tr}(\hat{A})| > 2$ corresponds to real eigenvalues with $\mu_1\mu_2 = 1$. The mapping is a hyperbolic rotation and is NOT stable. One of the solutions grows up with time. This is the parametrical resonance. Remember: in the last lecture we obtained same result.
- $|\mathrm{Tr}(\hat{A})| = 2$ is a condition for for calculation of the bounds of the stability. In this case $\mu_1 = \mu_2 = \pm 1$.

IV. EQUATION OF MATHIEU: PERTURBATION THEORY.

Now we go back to Mathieu's equation (1.1) and let us solve it by method of perturbation theory because of small parameter $h << 1$. At first we consider the case with $h = 0$:

$$\ddot{x} + \omega_0^2 x \; = \; 0. \tag{4.1}$$

The periodical solution is well known:

$$\begin{aligned} x(t) \; &= \; c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) \\ \dot{x}(t) \; &= \; c_1 \omega_0 \sin(\omega_0 t) + c_2 \omega_0 \cos(\omega_0 t), \end{aligned} \tag{4.2}$$

or in the matrix form:

$$\vec{z}(t) = \begin{pmatrix} \cos(\omega_0 t) & \sin(\omega_0 t) \\ -\omega_0 \sin(\omega_0 t) & \omega_0 \cos(\omega_0 t) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}. \tag{4.3}$$

Following to (3.2) we obtain equation for the monodromy matrix:

$$\vec{z}(T = \frac{2\pi}{\gamma}) = \begin{pmatrix} \cos(\frac{2\pi\omega_0}{\gamma}) & \sin(\frac{2\pi\omega_0}{\gamma}) \\ -\omega_0 \sin(\frac{2\pi\omega_0}{\gamma}) & \omega_0 \cos(\frac{2\pi\omega_0}{\gamma}) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \hat{A} \begin{pmatrix} 1 & 0 \\ 0 & \omega_0 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \tag{4.4}$$

where we look on the mapping for a period $T = \frac{2\pi}{\gamma}$, which is period of oscillation of frequency $\omega(t)$ (1.1). Check it yourself that the monodromy matrix $\hat{A}$ is

$$\hat{A}\left(\frac{2\pi}{\gamma}\right) = \begin{pmatrix} \cos(\frac{2\pi\omega_0}{\gamma}) & \frac{1}{\omega_0} \sin(\frac{2\pi\omega_0}{\gamma}) \\ -\omega_0 \sin(\frac{2\pi\omega_0}{\gamma}) & \cos(\frac{2\pi\omega_0}{\gamma}) \end{pmatrix} \tag{4.5}$$

Although we are considering the system with a periodical solution (when the perturbation is switched off $h = 0$), we can find the condition of appearance of the parametrical resonance $|\mathrm{Tr}(\hat{A})| = 2$:

$$\mathrm{Tr}(\hat{A}) \; = \; 2\cos\left(\frac{2\pi\omega_0}{\gamma}\right) = \pm 2, \tag{4.6}$$

$$\gamma \; = \; \frac{2\omega_0}{k}, \qquad k = 1, 2, 3... \tag{4.7}$$

We shall show that when we will switch on the perturbation (when the frequency of the system will be changing)the system leaves state of stability and becomes unstable. The parametrical resonance appears when γ is around $2\omega_0, \omega_0, \frac{1}{\omega_0}$... and this "around" depends on the parameter h. We will see it in the next lecture.