Lection 5.

I. CONSERVATION LAWS.

In this lecture we obtain the conservation laws for KdV equation. As we already know, the symmetries of the Lagrangian, according to Noether's theorem, corresponds to the conservation laws. But there are systems, which have conserved quantities which do not follow from invariance of the Lagrange function under some symmetry, but directly from the equation of motion. As you remember from the previous lecture, the Lagrange density for the KdV equation is

$$\mathfrak{L}_{KdV}(\dot{q}, \partial_x q, \partial_x^2 q) = \frac{(\partial_t q)(\partial_x q)}{2} - (\partial_x q)^3 - \frac{1}{2}(\partial_x^2 q)^2. \tag{1.1}$$

Using the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial \mathfrak{L}}{(\partial_t q)}\right) - \frac{\partial \mathfrak{L}}{\partial q} + \frac{d}{dx}\left(\frac{\partial \mathfrak{L}}{\partial(\partial_x q)}\right) - \frac{d^2}{dx^2}\left(\frac{\partial \mathfrak{L}}{\partial(\partial_x^2 q)}\right) = 0, \tag{1.2}$$

we can obtain the KdV equation as a EOM:

$$\partial_t r \ = 6r\partial_x r - \partial_x^3 r, \tag{1.3}$$

where $r(x,t) = \partial_x q(x,t)$. Let us find integrals of motion and show, if it is possible, that they correspond to the symmetries of the Lagrange function.

- The first conserved quantity for the KdV can be constructed directly from the equation of motion. Because of the right part of (1.3) is a full derivative, the quantity $I_0 = \int dx r(x,t)$ is a constant, indeed :

$$\frac{dI_0}{dt} = \int_{-\infty}^{+\infty} dx\ \partial_t r(x,t) = \int_{-\infty}^{+\infty} dx\ \partial_x(3r^2 - \partial_x^2 r) = (3r^2 - \partial_x^2 r)\Big|_{-\infty}^{+\infty} = 0. \tag{1.4}$$

This integral of motion agrees with invariance of Lagrangian under a shifting of field as a whole $q(x,t)' = q(x,t) + c$. The quantity $\int dx\ \partial_x q(x,t) = \int dx\ r(x,t)$ corresponds to this symmetry. $\boxed{\textit{Show this.}}$

- Next two conserved quantity are related to symmetries of the Lagrange function under translations in space and time:

$$\begin{aligned} x' &= x + a \\ t' &= t + t_0. \end{aligned}$$

In the Lagrange formalism we have a standard method how to construct corresponding conserved quantities:

$$\mathcal{P} = (\partial_x q)\frac{\partial \mathfrak{L}}{\partial(\partial_t q)} \qquad \rightarrow \qquad \mathcal{P}_{KdV} = \frac{1}{2}(\partial_x q)^2 \tag{1.5}$$

$$\mathcal{H} = (\partial_t q)\frac{\partial \mathfrak{L}}{\partial(\partial_t q)} - \mathfrak{L} \qquad \rightarrow \qquad \mathcal{H}_{KdV} = (\partial_x q)^3 + \frac{1}{2}(\partial_x^2 q)^2. \tag{1.6}$$

The conserved quantity for the spatial translation is the full momentum (1.5)

$$I_1 = \int_{-\infty}^{+\infty} dx \mathcal{P}_{KdV} = \frac{1}{2}\int_{-\infty}^{+\infty} dx \;\; r^2(x,t), \tag{1.7}$$

and for time translation is the full energy (1.6)

$$I_2 = \int_{-\infty}^{+\infty} dx \mathcal{H}_{KdV} = \int_{-\infty}^{+\infty} dx \;\; \left(r^3 + \frac{1}{2}(\partial_x r)^2\right). \tag{1.8}$$

- It can be shown directly from the EOM (1.3), that I_1 and I_2 are time-independent.

$\boxed{Show\;\; it\;\; yourself!}$

- Moreover, we can construct infinite number such quantities:

$$I_n = \int_{-\infty}^{+\infty} dx \;\; P_n(r, \partial_x^{(i)} r), \qquad\qquad P_n = \frac{dP_{n-1}}{dx} + \sum_{i=0}^{n-1} P_i P_{n-i-1}, \tag{1.9}$$

where the first iteration is $P_0 = r(x,t)$ and the second: $P_1 = \partial_x r(x,t) + r^2(x,t)$.
Using (1.3) and having the time enough, one can check, that I_n is a constant in the time for any n!!! These infinite number of integral of motion is NOT related to any symmetry of Lagrange function.
Remark: The set of I_n can be found with help of the quantum mechanics methods:
J.L. Lem , Introduction to soliton theory.
Problem: Show explicitly, that $\partial_t I_3 = 0$.

II. PARAMETRIC RESONANCE.

At first: what is the ordinary resonance? Let an external force acts on the oscillation system with eigenfrequency ω_0. And this force is periodic in time:

$$\ddot{x} + \omega_0^2 x = f\cos(\gamma t). \tag{2.1}$$

In this case we observe a changing of the original amplitude and it goes to infinity, when $\gamma \to \omega_0$. It is called a resonance.

Now let us consider the case, where originally permanent parameters of system are changing periodically (instead of the external force(2.1)):

$$\ddot{x} + \omega^2(t)x = 0, \qquad\qquad \omega(t+T) = \omega(t). \tag{2.2}$$

Such trick is very useful, when we don't know the external forces exactly. For instance, if we want to describe the moon motion (the three-body problem).
As a second order equation eq.(2.2)) has two linearly independent solutions $x_1(t)$ and $x_2(t)$.

From the periodicity of $\omega(t)$ we get, that $x_{1,2}(t+T)$ are also solutions of eq.(2.2)) and can be expressed as a linear combination of the first one:

$$
\begin{aligned}
x_1(t+T) &= \mu_1 x_1(t) + \nu_1 x_2(t); \\
x_2(t+T) &= \mu_2 x_1(t) + \nu_2 x_2(t).
\end{aligned}
$$

We always can choose two coefficients as zero and take the following ansazt:

$$
\begin{aligned}
x_1(t) &= \mu_1^{t/T}\Pi_1(t) \\
x_2(t) &= \mu_2^{t/T}\Pi_2(t),
\end{aligned} \tag{2.3}
$$

where $\Pi_{1,2}(t)$ are periodic functions. Substituting this ansazt into eq.(2.2)), we find a relation between μ_1 and μ_2:

$$
\begin{array}{l}
x_2 \cdot \left(\ddot{x_1} + \omega^2(t)x_1 = 0\right) \\
- \\
x_1 \cdot \left(\ddot{x_2} + \omega^2(t)x_2 = 0\right)
\end{array}
\quad \Rightarrow \quad \frac{d}{dt}\left(x_2\dot{x_1} - x_1\dot{x_2}\right) = 0, \tag{2.4}
$$

where the expression under the derivative is time-independent. In such way, using the ansazt (2.5)), we can write

$$
\begin{aligned}
x_2(t)\dot{x_1}(t) - x_1(t)\dot{x_2}(t) &= x_2(t+T)\dot{x_1}(t+T) - x_1(t+T)\dot{x_2}(t+T) \\
x_2(t)\dot{x_1}(t) - x_1(t)\dot{x_2}(t) &= \mu_1\mu_2 \cdot \left(x_2(t)\dot{x_1}(t) - x_1(t)\dot{x_2}(t)\right)
\end{aligned}
$$

or simply $\mu_1\mu_2 = 1$. It gives us the following solution(denote $\mu_1 = \mu_2 = \mu$):

$$
\begin{aligned}
x_1(t) &= \mu^{t/T}\Pi_1(t) \\
x_2(t) &= \mu^{-t/T}\Pi_2(t).
\end{aligned} \tag{2.5}
$$

So, we see, that if $\mu \neq 1$,one of solutions increases with time. This phenomenon is called Parametric Resonance.

III. PARAMETRIC RESONANCE: EQUATION OF MATHIEU.

Let us consider the system with defined expression for the frequency and try to define μ. Let the frequency be a simple periodic function of time:

$$
\ddot{x} + \omega_0^2(1 + h\cos(\gamma t))x = 0, \tag{3.1}
$$

where $h << 1$.
As we will see in the next lecture, the parametric resonance appears when $\gamma \to 2\omega_0$. Compare with the case (2.1)), when to get the resonance we must have the external periodic force with the frequency $\gamma \to \omega_0$.
Imagine if you are swinging. You are a oscillation system. If somebody swings you (the external force), he can act only one time per period. This is a resonance. But if you are swinging yourself, you are acting two times per period. This is exactly a parametric resonance!