Lection 4.

I. TWO-SOLITON SOLUTION.

On the previous lecture we obtained the one-soliton solution for KdV in the following form:

$$r_1(x-c_1t) = -\frac{c_1}{2}\cosh^{-2}\left(\frac{\sqrt{c_1}}{2}(x-c_1t)+y\right), \tag{1.1}$$

where y is initial (at $t=0$) position of soliton. The soliton is a pick with the amplitude $c_1/2$, which is moving as a whole with the speed $v=c_1$.

As well known for linear equations the sum of solutions is the solution. It is not true for our case, because the KdV equation is a non-linear equation. But it has the infinity set of solutions. Some of them are called multu-soliton solutions, because at large separation they degenerate to sum of one-soliton solutions.

The n-soliton solution is formulated in terms of matrixes and difficult for obtaining. Here we show the two-soliton solution explicitly:

$$r_2(x,t) \;=\; -\frac{(c_2-c_1)}{2}\frac{c_2\sinh^{-2}(\gamma_2)+c_1\cosh^{-2}(\gamma_1)}{(\sqrt{c_2}\mathrm{cth}(\gamma_2)-\sqrt{c_1}\mathrm{th}(\gamma_1))^2}\;,\quad \gamma_{1,2}=\frac{\sqrt{c_{1,2}}}{2}(x-c_{1,2}t)+y_{1,2} \tag{1.2}$$

$\boxed{Problem:\ Check,\ explicitly,\ that\ (1.2)\ is\ thesolution\ of\ KdV\ equation.}$

At large separation this solution is just a sum of two one-solitons. To show this let us take the $c_2>c_1$ and $t\gg 1$. Opening brackets we obtain:

$$r(x,t) \;=\; -\frac{c_2-c_1}{2}\Big[\frac{c_2}{c_2\mathrm{cosh}^2\gamma_2+c_1\mathrm{sinh}^2\gamma_2\mathrm{th}^2\gamma_1-2\sqrt{c_1c_2}\mathrm{cosh}\gamma_2\mathrm{sinh}\gamma_2\mathrm{th}\gamma_1}+$$
$$\frac{c_1}{c_2\mathrm{cosh}^2\gamma_1\mathrm{cth}^2\gamma_2+c_1\mathrm{sinh}^2\gamma_1-2\sqrt{c_1c_2}\mathrm{cosh}\gamma_1\mathrm{sinh}\gamma_1\mathrm{cth}\gamma_2}\Big]$$

At large t $\mathrm{sinh}\gamma\sim\mathrm{cosh}\gamma$ and $\mathrm{th}\gamma\sim 1$, so we have:

$$r(x,t) \;\sim\; -\frac{c_2-c_1}{2(\sqrt{c_2}-\sqrt{c_1})^2}\Big[\frac{c_2}{\mathrm{cosh}^2\gamma_2}+\frac{c_1}{\mathrm{cosh}^2\gamma_1}\Big]$$

One can recognize here the sum of two one-soliton solutions.

II. LAGRANGE FUNCTION FOR THE KDV EQUATION.

From the beginning we remind of the canonical form for the KdV equation

$$\partial_t r \;= 6r\partial_x r-\partial_x^3 r. \tag{2.1}$$

In this section we construct the Lagrange function and conserved quantities corresponding to the KdV equation as to a motion equation in terms of $q(x,t)=\partial_x^{-1}r(x,t)$:

$$\partial_t\partial_x q-6(\partial_x q)(\partial_x^2 q)+\partial_x^4 q=0. \tag{2.2}$$

So, as at the first lecture, we are starting with the Fermi-Pasta-Ulam chain of masses connected by springs. Let us write the action $S = \int_{t_1}^{t_2} dt L(q_i, t)$ for our chain, where the Lagrange function is a difference of the kinetic and potential energy of the all chain of masses:

$$S = \int dt \sum_i \left(\frac{m\dot{q}_i^2}{2} - \frac{K}{2}\left(q_{i+1} - q_i\right)^2 - \frac{\alpha}{3}\left(q_{i+1} - q_i\right)^3 \right). \tag{2.3}$$

Than we exchange the discrete index i to the continual x and work below with the generalized coordinate $q(x,t) \equiv q$ as a function of two variables. The sum over i in (2.3) turns to the integral over x and we get

$$S = \int_{t_1}^{t_2} dt \int_{x_1}^{x_2} dx\ \mathfrak{L}(\dot{q}, q, \partial_x q, \partial_x^2 q, \ldots), \tag{2.4}$$

where

$$\mathfrak{L}(\dot{q}, q, \partial_x q, \partial_x^2 q, \ldots) = \frac{m\dot{q}^2}{2} - \frac{K}{2}\left[e^{\partial_x} q - q\right]^2 - \frac{\alpha}{3}\left[e^{\partial_x} q - q\right]^3 \tag{2.5}$$

is the Lagrange density function for the FPU chain in the limit of big number of weights and a small distance between them. Here we should note, that the function $\mathfrak{L}$ is defined to accuracy of full derivatives over t and x. Such terms will be constants depending on $t_{1,2}$ and $x_{1,2}$ and will not contribute into the motion equations.

Now let us set aside the explicit expression for $\mathfrak{L}$ and try to construct its from equation of motion (2.2). We get the equation of motion in terms of the Lagrange density as a condition of extremum of the action (2.4)(the Variation Principle). For that we will assume q is a sum of the constant solution q_0 and a small variation: $q(x,t) = q_0 + \varepsilon\eta(x,t)$.

$$\begin{aligned} \frac{dS(q_0 + \varepsilon\eta)}{d\varepsilon} &= \int dt dx \frac{d}{d\varepsilon}\mathfrak{L}(\dot{q}, q, \partial_x q, \partial_x^2 q, \ldots) \\ &= \int dt dx \left(\frac{\partial \mathfrak{L}}{\partial \dot{q}}\dot{\eta} + \frac{\partial \mathfrak{L}}{\partial q}\eta + \frac{\partial \mathfrak{L}}{\partial(\partial_x q)}\eta_x + \frac{\partial \mathfrak{L}}{\partial(\partial_x^2 q)}\eta_{xx} + \ldots \right) \equiv 0 \end{aligned} \tag{2.6}$$

Integrating by parts and neglecting by full derivatives we get the equation of motion for any non-linear Lagrange density

$$\frac{d}{dt}\left(\frac{\partial \mathfrak{L}}{(\partial_t q)}\right) - \frac{\partial \mathfrak{L}}{\partial q} + \frac{d}{dx}\left(\frac{\partial \mathfrak{L}}{\partial(\partial_x q)}\right) - \frac{d^2}{dx^2}\left(\frac{\partial \mathfrak{L}}{\partial(\partial_x^2 q)}\right) + \ldots = 0. \tag{2.7}$$

It is easy to check by a substitution, that Eq.(2.7) coincides with Eq.(2.2), when

$$\mathfrak{L}_{KdV}(\dot{q}, \partial_x q, \partial_x^2 q) = \frac{(\partial_t q)(\partial_x q)}{2} - (\partial_x q)^3 - \frac{1}{2}(\partial_x^2 q)^2. \tag{2.8}$$

Note also, that the Lagrangian does not depend on the $q(x,t)$, but only on its derivatives.

Now let us go back to the explicit expression for Lagrange density (2.9) and, making approximations like in the first and second lectures($\partial_x q \ll q,\ \alpha \ll K$), obtain corresponding Lagrangian. As you remember we assumed $q = q(x - ct, t)$. So (2.9) gives us

$$\mathfrak{L}(\dot{q}, q, \partial_x q, \partial_x^2 q, \ldots) = \frac{m}{2}(q_t - c q_x)^2 - \frac{K}{2}(\partial_x q + \frac{\partial_x^2 q}{2} + \frac{\partial_x^3 q}{6} + \ldots)^2 - \frac{\alpha}{3}(\partial_x q + \ldots)^3. \tag{2.9}$$

The first non-zero term of expansion is the wave Lagrange density $\mathfrak{L}_{wave}$, which corresponds to the wave equation under EOM (2.7):

$$\mathfrak{L}_{wave} = \frac{m}{2}(\dot{q})^2 - \frac{K}{2}(q_x)^2 \qquad \Rightarrow \qquad m\ddot{q} = K q_{xx}. \tag{2.10}$$

The next term is a full derivative $\frac{K}{4}\partial_x(q_x)^2$ and should be excluded, but the next-to-next one gives a nontrivial result:

$$\begin{aligned}\mathfrak{L}(\dot{q}, \partial_x q, \partial_x^2 q) &= \frac{mc^2(q_x)^2}{2} - cmq_t q_x - \frac{K}{2}\left[(\partial_x q)^2 + \frac{(\partial_x^2 q)^2}{4} + \frac{(\partial_x q)(\partial_x^3 q)}{3}\right] - \frac{\alpha}{3}(\partial_x q)^3 \\ &= -cmq_t q_x + \frac{mc^2}{2}\frac{(\partial_x^2 q)^2}{12} - \frac{\alpha}{3}(\partial_x q)^3, \end{aligned} \tag{2.11}$$

where we have used $K = mc^2$ and neglected q_t^2. Again, according to EOM (2.7), we obtain

$$-2c\partial_x \dot{q} = \frac{c^2}{12}\partial_x^4 q + \frac{2\alpha}{m}\partial_x q \partial_x^2 q \tag{2.12}$$

or with $\partial_x q = r(x,t)$

$$-2cr_t(x,t) = \frac{c^2}{12}\partial_x^3 r(x,t) + \frac{2\alpha}{m} r(x,t)\partial_x r(x,t). \tag{2.13}$$

This is nothing else but the KdV equation, which after rescaling (see L-3) takes a form (2.1).

Problem: Make sure that after rescaling Lagrangian (2.11) *is equal to* (2.8).