Lection 3

I. THE CLASSICAL EXPRESSION FOR THE KDV EQUATION.

On the last lecture we obtained following expression for KdV:

$$-2c\partial_t r(x,t) \;=\; \frac{c^2}{12}\partial_x^3 r(x,t) + \frac{2\alpha}{m} r(x,t)\partial_x r(x,t). \tag{1.1}$$

To get the classical form of KdV equation

$$\partial_t r \;= 6r\partial_x r - \partial_x^3 r, \tag{1.2}$$

we rescale variables $t' = \tau t$, $x' = \xi x$, $r' = \rho r(x' - ct')$ in Eq.(1.1). In this way we obtain a system of equations with a simple solution:

$$\begin{aligned} \frac{\tau}{\rho} &= -2c, & \tau &= \left(\frac{72m^3c^7}{\alpha^3}\right)^{1/5} \\ \frac{\xi}{\rho^2} &= \frac{\alpha}{3m}, & \xi &= \left(\frac{mc^4}{48\alpha}\right)^{1/5} \\ \frac{\xi^3}{\rho} &= -\frac{c^2}{12}, & \rho &= -\left(\frac{9c^2m^3}{4\alpha^3}\right)^{1/5}. \end{aligned}$$

II. ONE-SOLITON SOLUTION FOR THE KDV EQUATION.

To solve (1.2) equation we use the ansazt $r(x,t) = r(x - c_1 t)$. So, the equation depends on only one variable x:

$$-c_1\partial_x r(x) = 3\partial_x\left(r^2(x)\right) - \partial_x^3 r(x).$$

It is easy to see, that after integration over x we arrive at

$$\frac{d^2}{d\xi^2} r(\xi) = 3r^2(\xi) + c_1 r(\xi) + \omega$$

and this expression is nothing else but the Newton equation for some physical point with $m = 1$ in a cube potential:

$$\ddot{\phi} = 3\phi^2 + c_1\phi + \omega = -\frac{dV(\phi)}{d\phi},$$

where we assume, that $r(\xi) = r(x - c_1 t)$ plays a role of generalized coordinate ϕ, and ξ - a role of the time. The potential has a form (Let us assume the integration constant ω equals zero!)

$$V(\phi) = -\phi^3 - \frac{c_1}{2}\phi^2 - \omega\phi = -\phi^2(\phi + \frac{c_1}{2})$$

and allows a finite motion (the function $r(x - ct)$ is finite), if the total energy of our system is negative $E \leq 0$ (see Fig.2(a)). Let us consider the case $E = 0$. The solution of equation of motion is

$$\xi(\phi) = \int \frac{d\phi}{\sqrt{2(E - V(\phi))}} = \frac{-1}{\sqrt{c_1}} \int \frac{d\phi'}{\phi'\sqrt{-\frac{2}{c_1}\phi' + 1}},$$

where we changed the variable of integration $\phi \to -\phi'$, because we are interesting in finite motion, which corresponds to negative ϕ.

Let us calculate this integral, using the substitution $\frac{2}{c_1}\phi' = \sin^2(\alpha)$:

$$\xi(\phi) = \frac{-1}{\sqrt{c_1}} \int \frac{d\phi'}{\phi'\sqrt{-\frac{2}{c_1}\phi' + 1}} = \frac{-2}{\sqrt{c_1}} \int \frac{d\alpha}{\sin(\alpha)} = \frac{2}{\sqrt{c_1}} \int \frac{d\cos(\alpha)}{(1-\cos^2(\alpha))}.$$

Then assuming that $\cos(\alpha) = t$, we have

$$Int = \frac{2}{\sqrt{c_1}} \int \frac{dt}{(1-t)(1+t)} = \frac{1}{\sqrt{c_1}} \int dt \left(\frac{1}{(1-t)} + \frac{1}{(1+t)}\right) = \frac{1}{\sqrt{c_1}} \ln\left[\frac{1+t}{1-t}\right] + const.$$

Using the expression for hyperbolic tangent $\tanh^{-1}(t) = \frac{1}{2}(\ln(1+t) - \ln(1-t))$ and restoring the argument $\frac{2}{c_1}\phi' = \sin^2(\alpha) = 1 - t^2$ and $\Rightarrow t = \pm\sqrt{1 - \frac{2\phi'}{c_1}} = \pm\sqrt{1 + \frac{2\phi}{c_1}}$, we get

$$\xi(\phi) = \frac{2}{\sqrt{c_1}} \tanh^{-1}\left(\pm\sqrt{1 + \frac{2\phi}{c_1}}\right)$$

or, taking into account a parity of function $\tanh(t)$,

$$\phi(\xi) = \frac{c_1}{2}\left[\tanh^2\left(\frac{\sqrt{c_1}}{2}\xi\right) - 1\right] = -\frac{c_1}{2}\cosh^{-2}\left(\frac{\sqrt{c_1}}{2}\xi\right).$$

So, we have got the localized in space soliton solution of the KdV equation:

$$r(x - c_1 t) = -\frac{c_1}{2}\cosh^{-2}\left(\frac{\sqrt{c_1}}{2}(x - c_1 t)\right).$$

PROBLEM 1: Calculate the integral $I_n(c_1) = \int_{-\infty}^{\infty} dx (r(x))^n$.

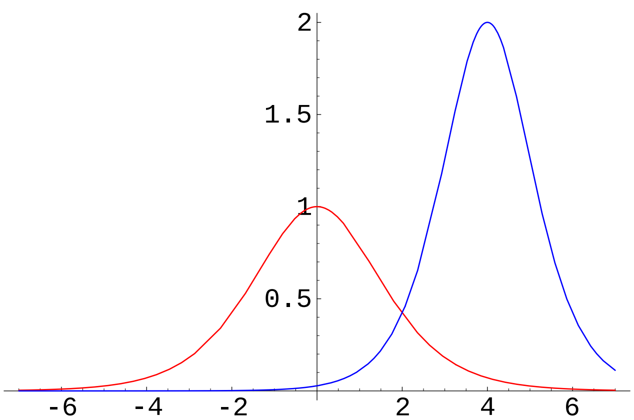

FIG. 1: Blue soliton moves faster than Red one!

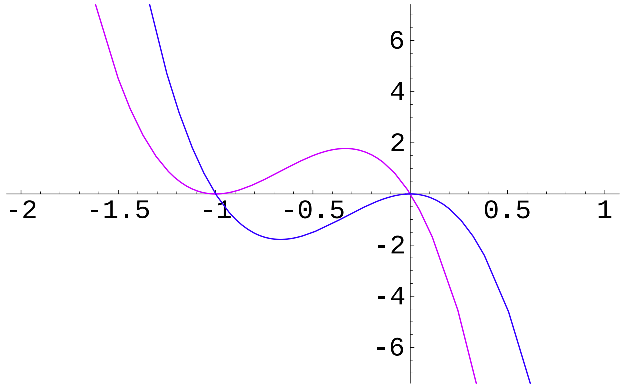

FIG. 2: The blue potential (a) with a soliton solution; the lilac potential (b) with a oscillation.

PROBLEM 2: Let us go back to the general form for our potential

$$V(\phi) = -\phi(\phi^2 + \frac{c}{2}\phi + \omega) = -\phi(\phi - \phi_1)(\phi - \phi_2),$$

where the roots of the quadratic equation are $\phi_{1,2} = 1/4\left(-c \pm \sqrt{c^2 - 16\omega}\right)$. The case with $\omega = 0$ was considered above and gave us the one-soliton solution. Please, investigate a possible finite motion at $E = 0$, especially consider the limiting case of $(\omega \to c^2/16)$.