I. SOLUTION OF THE SECOND PROBLEM (HOMEWORK)

Find an asymptotic at $\alpha \to 0$ of $\sum_{n=0}^{\infty} n e^{-\alpha n^2}$.

To find the asymptotic we will use the following trick. It is well known that

$$e^{a\partial_x} f(x) \;=\; f(x+a)$$

We rewrite:

$$\lim_{\alpha\to 0}\sum_{n=0}^{\infty} n e^{-\alpha n^2} \;=\; \lim_{\alpha\to 0}\sum_{n=0}^{\infty} e^{n\partial_x} x e^{-\alpha x^2}\Big|_{x=0} \;=\; \lim_{\alpha\to 0}\frac{1}{1-e^{\partial_x}} x e^{-\alpha x^2}\Big|_{x=0}$$

where we use the geometric progression for summing the series over n. Next, we expand the prefactor in powers of derivatives and acting them on function

$$\lim_{\alpha\to 0}\frac{1}{1-e^{\partial_x}} x e^{-\alpha x^2}\Big|_{x=0} \;=\; \lim_{\alpha\to 0}\Big[-\partial_x^{-1}+\frac{1}{2}+..\Big] x e^{-\alpha x^2}\Big|_{x=0} \;=\; \lim_{\alpha\to 0}\frac{1}{2\alpha}e^{-\alpha x^2}+\frac{x}{2}\Big|_{x=0}+..$$

where we use that $\partial_x^{-1} = \int dx$. One can easy see that higher terms are finite in limit $\alpha \to 0$. Finite answer is following:

$$\lim_{\alpha\to 0}\sum_{n=0}^{\infty} n e^{-\alpha n^2} \sim \frac{1}{2\alpha} + \text{const} + ..$$

Find the next term for asymptotic expansion!

II. THE KORTEWEG DE VRIES EQUATION

On the previous lecture (see *http://www.tp2.rub.de/ maximp/learning/seminar1.pdf*) we discuss the infinity row of weights connected by nonlinear spring. And we obtain that the Newton law takes a form:

$$m\ddot{q} \;=\; -K\Big[2q - e^{\partial_x}q - e^{-\partial_x}q\Big] + \alpha\Big\{\Big[(e^{\partial_x}-1)q\Big]^2 - \Big[(e^{-\partial_x}-1)q\Big]^2\Big\}$$

where $q \equiv q(x,t)$, the function deviation of x's weight from the point of equilibrium.

The nonlinearity of the springs is very small $\alpha \ll K$. Also we wish to consider that the function q varies very slowly (the deviation from equilibrium is small), $\partial_x q \ll q$. That is why we can expand exponents into series. In the first lecture we find the first approximation – a usual wave equation. Now let us obtain the next iteration:

$$m\ddot{q} \;=\; K\Big[\partial_x^2 q + \frac{1}{12}\partial_x^4 q\Big] + 2\alpha(\partial_x q)(\partial_x^2 q)$$

We know that without additional perturbation the solution was $q = q(x-ct)$. Now it is not so: q is arbitrary function of x and t. So we can say that $q = q(x-ct,t)$. But the dependence of the second argument t is slow, because it is generated by the small perturbation in our equation. Such ansatz allows us to write:

$$\ddot{q} \;=\; c^2\partial_x^2 q(x-ct,t) - 2c\partial_x\dot{q}(x-ct,t)$$

where the dot acts on the second variable. We neglect $\ddot{q}(x-ct,t)$ because it much smaller than our accuracy.

Substituting this into equation we obtain:

$$-2c\partial_x\dot{q} \;=\; \frac{c^2}{12}\partial_x^4 q + \frac{2\alpha}{m}\partial_x q \partial_x^2 q$$

Introducing the function $r(x,t) = \partial_x q$ we obtain the equation on it:

$$-2c\partial_t r(x,t) \;=\; \frac{c^2}{12}\partial_x^3 r(x,t) + \frac{2\alpha}{m} r(x,t)\partial_x r(x,t)$$

This is a form of the famous Korteweg de Vries equation.

Actually the classical view of KdV equation is following:

$$\partial_t r \;= 6r\partial_x r - \partial_x^3 r$$

But we can turn to this view after some redefenition of x and t.

Problem 1. How should we to redefine x *and* t?

On the next lecture we will obtain the one-soliton solution of KdV equation. We will discuss the equations of motion for KdV systems.

Problem 2. Try to find any solution of KdVequation. Hint. Using ansatz $r = r(x - c_1 t)$ and integrating out one derivative, one would obtain usual differential equation.