# Supersymmetry in quantum mechanic, lec.1

PACS numbers:

The supersymmetry (SUSY) is the symmetry between bosons and fermions. In order to describe it we want to introduce the very convenient in quantum mechanic(QM) technique – the creation and annihilation operators.

Let us consider the simplest task in QM: the harmonic oscillation. The potential of harmonic oscillation is

$$V(x) = \frac{1}{2}x^2,$$

for simplicity we put all dimensional parameters $\hbar = \omega = m = 1$. In such notation the canonical commutator is simply $[x, \hat{p}] = i$. The Schrodinger equation has a form

$$\hat{H}\psi = E\psi, \quad \hat{H} = \frac{1}{2}(\hat{p}^2 + x^2). \tag{1}$$

Introducing the operators

$$a^{\pm} = \frac{1}{\sqrt{2}}(x \mp i\hat{p}),$$

with properties

$$(a^+)^\dagger = a^-, \quad [a^-, a^+] = 1, \qquad\qquad \text{(check it!)}$$

one finds that Hamiltonian can be rewritten as

$$\hat{H} = a^+a^- + \frac{1}{2}.$$

Let us investigate the action of the operators $a^{\pm}$ on the state $|\psi\rangle$. The energy of the state $a^{\pm}|\psi\rangle$ can be found with help of Hamiltonian

$$\hat{H}a^{\pm}|\psi\rangle = a^{\pm}\hat{H}|\psi\rangle + [\hat{H}, a^{\pm}]|\psi\rangle = (E \pm 1)a^+|\psi\rangle.$$

Thus the operator $a^{+(-)}$ increase (decrease) the energy of state by one quant. For this reason this operators are called creation (annihilation) operators.

Let us consider the lowest energy state, which is usually called vacuum and denote as $|0\rangle$. Because this is the lowest state the action of annihilation operator gives zero

$$a^-|0\rangle = 0.$$

This property allows one obtain the energy of this state

$$\hat{H}|0\rangle = \frac{1}{2}|0\rangle. \tag{2}$$

The next excited state with energy $\frac{3}{2}$ can be obtained by acting of the $a^+$

$$a^+|0\rangle = |1\rangle, \quad \hat{H}|1\rangle = \frac{3}{2}|1\rangle.$$

Now, it is easy to find that

$$(a^+)^n|0\rangle \sim |n\rangle, \quad \hat{H}|n\rangle = \big(n + \frac{1}{2}\big)|n\rangle, \tag{3}$$

where on the right-hand-side we obtain well-known oscillator spectrum. Note that it follows that the number of bosons in system is given by operator $a^+a^-$.

Considered operators are creation and annihilation operators for bosons. Now we want to introduce similar operators for the fermions. The main difference between fermions and bosons is that one satisfies the Fermi-Dirac statistic and another the Bose-Einstein statistic. The difference in this statistics goes back to the Pauli principle, which postulates that in the system can not be two fermions with the same quantum numbers. We already interpreted the state $|n\rangle$ as $n$ bosonic excitation. For the fermionic excitations the only possible states are $|0_F\rangle$ and $|1_F\rangle$ (the subscript $F$ stays here in order to make difference between fermion and boson states), because $|2_F\rangle$ and higher are forbidden by Pauli principle.

We introduce the creation and annihilation operators for the fermions with the properties

$$(f^+)^\dagger = f^-, \quad f^-|0_F\rangle = 0, \quad f^+|1_F\rangle.$$

From that follows that $(f^-)^2 = (f^+)^2 = 0$. Operators with this property are called nilpotent. The anticommutator of $f^\pm$ is

$$\{f^+, f^-\} = f^+f^- + f^-f^+ = 1,$$

which can be proved by acting on the arbitrary fermionic state (these are only two states, $|0_F\rangle$ and $|1_F\rangle$).

Let us combine all obtained information on the creation and annihilation operators into one table

$$(a^\pm)^\dagger = a^\mp \qquad (f^\pm)^\dagger = f^\mp \qquad (f^+)^2 = (f^-)^2 = 0$$

$$a^+|n\rangle = (n+1)|n+1\rangle \qquad a^-|n\rangle = |n-1\rangle \qquad f^\pm|n_F\rangle = |(n\pm 1)_F\rangle$$

$$[a^-, a^+] = 1 \qquad \{f^+, f^-\} = 1 \qquad [a, f] = 0$$

The SUSY is the symmetry which turns the bosons to fermions and back. Let us define the SUSY operators $Q^\pm$ as

$$Q^+|n, k_F\rangle = |n-1, (k+1)_F\rangle, \quad Q^-|n, k_F\rangle = |n+1, (k-1)_F\rangle. \tag{4}$$

It easy to build such operators from creation and annihilation operators

$$Q^\pm = q\ a^\mp f^\pm.$$

Here $q$ is arbitrary real constant. These operators are not Hermitian, $(Q^\pm)^\dagger = Q^\mp$. But one could build two linear Hermitian combinations

$$Q_1 = Q^+ + Q^-, \quad Q_2 = -i(Q_1 - Q_2).$$

First of all we note that the squares of this operators are equal to each other and they are the operators of number of particles in the system

$$Q_1^2 = Q_2^2 = q^2\ (a^+a^- + f^+f^-). \qquad (\textit{Check it!})$$

The operators $Q_1$ and $Q_2$ anti-commute with each other

$$\{Q_1, Q_2\} = 0, \qquad\qquad (\textit{Check it!!})$$

We want to build the Hamiltonian for the SUSY system. It should satisfy two properties, 1. It should be Hermitian 2. It should commute with $Q_1$ and $Q_2$. The natural choice is

$$\hat{H} = Q_1^2.$$

Together with the operators $Q_i$ it forms the SUSY algebra

$$[\hat{H}, Q_i] = 0, \quad \{Q_i, Q_j\} = 2\delta_{ij}\hat{H}. \qquad (\textit{Check it!!})$$