MULTIPLE INTEGRALS.

1. Find $F'(\alpha)$, if

$$F(\alpha) = \int_0^{\alpha} \frac{\ln(1+\alpha x)}{x} dx,$$
$$F(\alpha) = \int_0^{\alpha} f(x+\alpha, x-\alpha)\ dx,$$
$$F(\alpha) = \int_0^{\alpha^2} dx \int_{x-\alpha}^{x+\alpha} \sin(x^2+y^2-\alpha^2)\ dy.$$

2. Find $F''(x)$, if

$$F(x) = \frac{1}{h^2}\int_0^h d\xi \int_0^h f(x+\xi+y)\ dy, \qquad h>0.$$

3. Using

$$\frac{\arctan x}{x} = \int_0^1 \frac{dy}{1+x^2y^2},$$

compute integral

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} \frac{\arctan x}{x}.$$

4. Using

$$\frac{e^{-ax}-e^{-bx}}{x} = \int_a^b e^{-xy} dy,$$

compute integral

$$\int_0^{\infty} \frac{e^{-ax}-e^{-bx}}{x} dx.$$

5. Proof Frullani formula

$$\int_0^{\infty} \frac{f(ax)-f(bx)}{x} dx = (f(0)-f(\infty)) \ln\left(\frac{b}{a}\right).$$

6.

$$\int_0^{\infty} \frac{e^{-ax}-e^{-bx}}{x} \sin mx\ dx.$$

7. Compute Gauss integral (hint: consider I^2)

$$I = \int_0^{\infty} e^{-ax^2}\ dx.$$

8. Compute Dirichlet integral

$$D(\beta) = \int_0^{\infty} \frac{\sin \beta x}{x}\ dx,$$

hint: use $I(\alpha,\beta) = \int_0^{\infty} e^{-\alpha x} \frac{\sin\beta x}{x}\ dx$ and $I'(\alpha,\beta)$.

9. Compute Laplace integral

$$L(\alpha) = \int_0^{\infty} \frac{\cos \alpha x}{1+x^2}\ dx,$$

hint: use $(1+x^2)^{-1} = \int\limits_0^{\infty} dy\ e^{-y(1+x^2)}$.

10.

$$\int_{-\infty}^{\infty} \frac{\cos \alpha x}{ax^2 + bx + c} \, dx, \quad a > 0, \quad ac - b^2 > 0.$$

11. Using Fresnel integral (finish its calculations we started on seminar)

$$\int_0^\infty \sin x^2 \, dx,$$

calculate following

$$\int_{-\infty}^{\infty} \cos x^2 \cos(ax) \, dx.$$

12. Reduce triple integral

$$\int_0^x d\xi \int_0^\xi d\eta \int_0^\eta d\tau \quad f(\tau)$$

to 1-dimensional integral.

13.

$$\iint\limits_{x+y \geq 1, \quad 0 \leq x \leq 1} \frac{dxdy}{(x+y)^p}.$$

14.

$$\iint\limits_{|y| \leq x^2, \quad x^2+y^2 \leq 1} \frac{dxdy}{x^2+y^2}.$$

15.

$$\iint\limits_{x^2+y^2 \leq 1} dxdy \ln \left(\frac{1}{\sqrt{x^2+y^2}} \right).$$

16. Proof that

$$\int_0^t dt_1 \int_0^{t_1} dt_2 ... \int_0^{t_{n-1}} dt_n f(t_1) f(t_2) ... f(t_n) = \frac{1}{n!} \left\{ \int_0^t d\tau \ f(\tau) \right\}^n.$$

17. Proof that

$$\int_0^t dt_1 \int_0^{t_1} dt_2 ... \int_0^{t_{n-1}} dt_n f(t_n) = \int_0^t d\tau \ f(\tau) \frac{(t-\tau)^{n-1}}{(n-1)!}.$$

18. Change order of integration:

$$\int_0^{2\pi} dx \int_0^{\sin x} dy \ f(x,y).$$

19. Find volume restricted by the surface (use spherical coordinates)

$$(x^2 + y^2 + z^2)^3 = 3xyz.$$

20.

$$\iiint\limits_{x^2+y^2+z^2 \leq 1} dxdydz \quad x^m y^n z^p.$$

21.

$$\iint\limits_{\substack{0\le x\le\pi,\\ 0\le y\le\pi}} dxdy|\cos(x+y)|.$$

22. Rewrite the integral in new variables $u = x$, $v = y/x$:

$$\int\limits_a^b dx \int\limits_{\alpha x}^{\beta x} f(x,y)\ dy.$$

23. Calculate the line integral

$$\int\limits_C y^2 ds,$$

where C is cycloid

$$\begin{cases} x = a(t-\sin t) \\ y = a(1-\cos t) \end{cases},\quad 0\le t\le 2\pi\ .$$

24. Calculate the line integral

$$\int\limits_C \frac{1}{y^2} ds,$$

where C is Catenoid $y = a\cosh\left(\frac{x}{a}\right)$.

25. Calculate the line integral

$$\int\limits_C \left[(x^2+y^2)dx + (x^2-y^2)dy\right],$$

where C is $y = 1-|1-x|,\quad 0\le x\le 2.$

26.

$$\oint\limits_R \frac{(x+y)dx-(x-y)dy}{x^2+y^2},$$

where R is circle $x^2+y^2=a^2$.

27. Using Green formula, compute

$$\oint\limits_{x^2+y^2=a^2} \left[(xy^2\ dy - x^2y\ dx\right],$$

28.

$$\oint\limits_{x^2+y^2=a^2} e^{-(x^2-y^2)}\left[\sin(2xy)\ dy + \cos(2xy)\ dx\right],$$

29. Consider two cases for the integral with simple closed contour C:

$$\oint\limits_C \frac{x\ dy - y\ dx}{x^2+y^2},$$

a) Point (0,0) is inside of contour;

b) Point (0,0) is outside of contour.

30. Find area enclosed by the astroid:

$$\begin{cases} x = a\cos^3 t \\ y = b\sin^3 t \end{cases}, \quad 0 \le t \le 2\pi$$

hint: use that according to Green formula area is given by

$$S = \frac{1}{2} \oint\limits_C (x\ dy - y\ dx).$$

31. Compute

$$u(x, y) = \frac{1}{2} \oint\limits_{\xi^2+\eta^2=R^2} \ln\left(\frac{1}{r}\right), \qquad r = \sqrt{(\xi - x)^2 + (\eta - y)^2}.$$

32. Find functions $P(x, y)$ and $Q(x, y)$ such that the integral

$$I = \oint\limits_C [P(x + \alpha, y + \beta)dx + Q(x + \alpha, y + \beta)dy],$$

for any contour C was independent of constant α and β.

33. Proof Green formula.