Dispersive analysis of isospin breaking in $K_{\ell 4}$ decays

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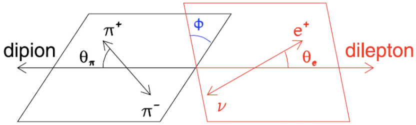
Bad Honnef workshop - February 13 2011

$\pi\pi$ scattering as a test of $N_f = 2$ χPT

$\pi\pi$ scattering

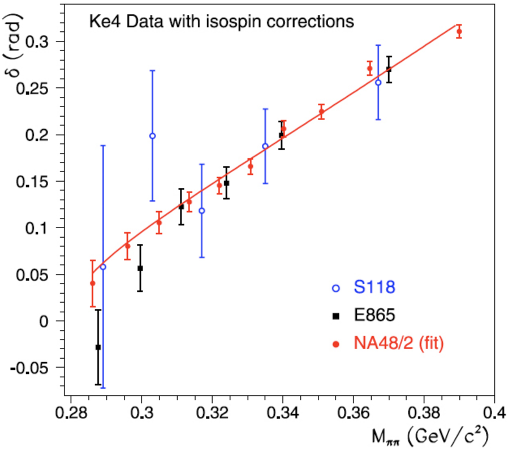
- The best test of $N_f = 2$ χPT ($m_{u,d} \to 0$, m_s physical)
- Tested experimentally through $\pi\pi$ (re)scattering
- Pionic atoms, cusp in $K \to 3\pi$, $K_{\ell 4}$

2010: final analysis of high-precision data from NA48/2 $K^\pm \to \pi^+\pi^-\ell^\pm\nu$



- Angular analysis: interference between S and P waves
- In isospin limit, related to phase shift difference $\delta_0^0 - \delta_1^1$ (δ_ℓ^I) of $\pi\pi$ scattering amplitude
- which can be compared to a dispersive representation (Roy equations) to extract scattering lengths a_0^0 and a_0^2
- to be compared to predictions from $N_f = 2$ χPT

Phase shifts

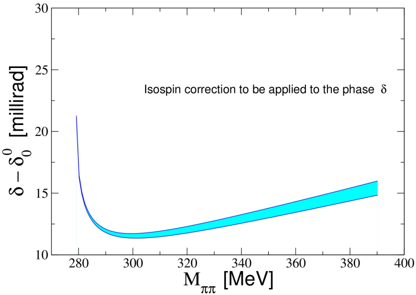


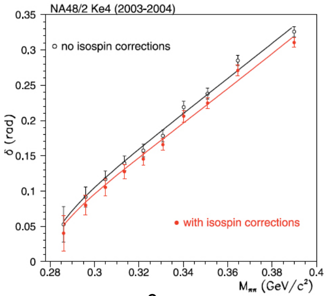
[Eur.Phys.J.C70:635-657,2010]

Gasser & Rusetsky : at this level of accuracy, isospin breaking matters

Experiment treat real & virtual photons, but not mass effects

Isospin-breaking corrections





- Mass differences : phase space, $M_{\pi^+} \neq M_{\pi^0}$, π^0-η mixing in loops
- Difference between δ_S in K_{e4} and δ_0^0 in isospin limit estimated using one-loop χPT [Colangelo, Gasser, Rusetsky]

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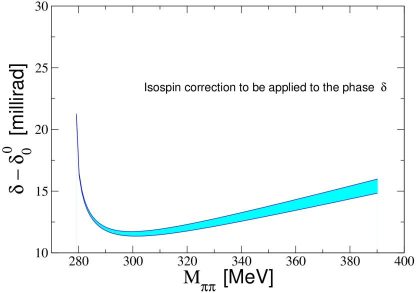

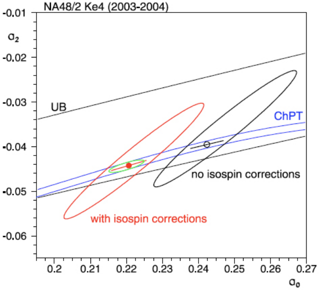

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- But computation in χPT corresponding to a particular (a_0^0, a_0^2)

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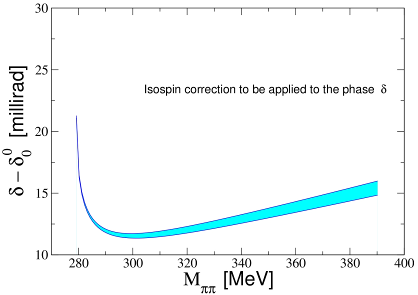

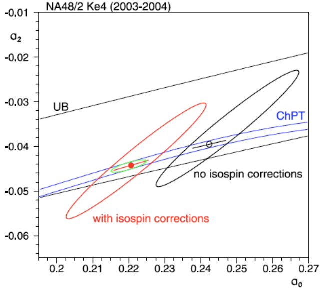

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- Difference between δ_S in K_{e4} and δ_0^0 in isospin limit estimated using one-loop χPT [Colangelo, Gasser, Rusetsky]
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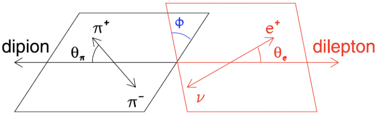
Isospin breaking for arbitrary (a_0^0, a_0^2) by dispersive approach [V. Bernard, SDG, M. Knecht, in preparation]

Isospin breaking in $K_{\ell 4}$ decays

$K_{\ell 4}$ form factors



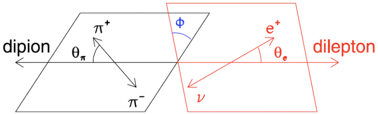
$$\langle \pi^a(p_a)\pi^b(p_b)|iA_\mu^{4-i5}|K(k)\rangle \times M_K$$
$$= F^{ab}(p_a+p_b)_\mu + G^{ab}(p_a-p_b)_\mu$$
$$+R^{ab}(k-p_a-p_b)_\mu$$

- Form factors fncts of $s=(p_a+p_b)^2, t=(k-p_a)^2, u=(k-p_b)^2$

$$s+t+u = M_a^2+M_b^2+M_K^2+s_l$$

with inv hadronic mass s and leptonic mass $s_l=(k-p_a-p_b)^2$

$K_{\ell 4}$ form factors



$$\langle \pi^a(p_a)\pi^b(p_b)|iA_\mu^{4-i5}|K(k)\rangle \times M_K = F^{ab}(p_a+p_b)_\mu + G^{ab}(p_a-p_b)_\mu + R^{ab}(k-p_a-p_b)_\mu$$

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- Linear combinations with "interesting" properties w.r.t unitarity:

$$\mathcal{F}^{ab} = F^{ab} + \left[\frac{M_a^2-M_b^2}{s} + \frac{M_K^2-s-s_l}{s}\sqrt{\frac{\lambda_{ab}}{\lambda_{K\ell}}}\cos\theta_\pi\right] G^{ab}$$

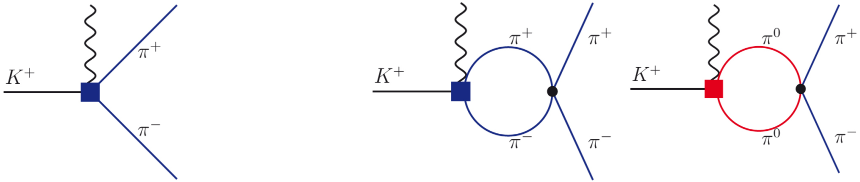
$$\mathcal{G}^{ab} = G^{ab} \qquad \lambda_{ab} = [s-(M_a-M_b)^2][s-(M_a+M_b)^2]$$

$$\mathcal{R}^{ab} = R^{ab} + \ldots \times F^{ab} + \ldots \times G^{ab}$$

  $(\mathcal{F},\mathcal{G})$ mix under crossing symmetry, but not with $\mathcal{R}$ (discarded)

Unitarity outside the isospin limit

$$\mathcal{F}^{ab} = \sum_{\ell \geq 0} f_\ell^{ab}(s, s_l) P_\ell(\cos\theta_\pi) \qquad \mathcal{G}^{ab} = \sum_{\ell \geq 1} g_\ell^{ab}(s, s_l) P'_\ell(\cos\theta_\pi)$$



$$\mathrm{Im}\, f_\ell^{+-}(s, s_l) = \mathrm{Re}\left\{ \frac{\sqrt{\lambda_{+-}}}{s} t_\ell^{+-+-}(s) f_\ell^{+-*}(s, s_l) + \frac{\sqrt{\lambda_{00}}}{2s} t_\ell^{00+-}(s) f_\ell^{00*}(s, s_l) \right\} + \mathcal{O}(E^8)$$

involving $\pi\pi$ scattering amplitudes $\pi^a\pi^b \to \pi^+\pi^-$

$$A^{ab+-}(s, t) = 16\pi \sum_\ell (2l + 1) t_\ell^{ab+-}(s) P_\ell(\cos\theta)$$

as well as form factors $f_\ell^{ab}(s, s_l)$

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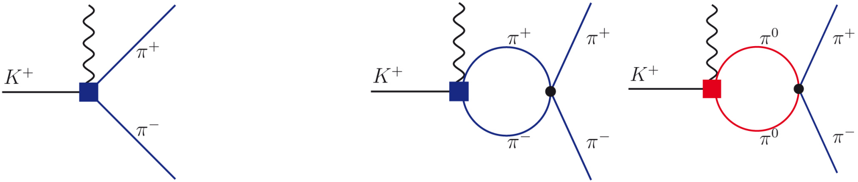

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$$\mathrm{Im}\, g_\ell^{+-}(s, s_l) = \mathrm{Re}\left\{ \frac{\sqrt{\lambda_{+-}}}{s} t_\ell^{+-+-}(s) g_\ell^{+-*}(s, s_l) + \frac{\lambda_{+-}}{2\sqrt{\lambda_{00}}} t_\ell^{00+-}(s) g_\ell^{00*}(s, s_l) \right\} + \mathcal{O}(E^8)$$

Unitarity outside the isospin limit

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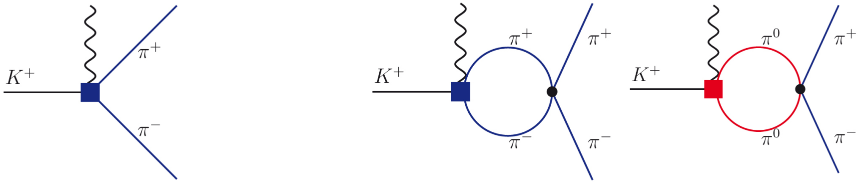

$$\mathrm{Im}\, f_\ell^{+-}(s, s_l) = \mathrm{Re}\Big\{ \frac{\sqrt{\lambda_{+-}}}{s} t_\ell^{+-+-}(s) f_\ell^{+-*}(s, s_l) + \frac{\sqrt{\lambda_{00}}}{2s} t_\ell^{00+-}(s) f_\ell^{00*}(s, s_l) \Big\} + \mathcal{O}(E^8)$$

- Phase $f^{+-}(s, s_l) \neq$ phase $t_\ell^{+-+-}(s)$, and depends on s and s_l
- Isospin breaking from phase space ($\lambda_{+-} \neq \lambda_{00}$),
  scattering ($t_\ell^{+-+-} \neq t_\ell^{00+-}$) and form factors ($f_\ell^{+-} \neq f_\ell^{00}$)

Chiral counting and phase shifts

- For S and P-waves, up to $\mathcal{O}(E^6)$

$$f^{ab} = \bar{f}^{ab}\Big|_{[E^0]} + \hat{f}^{ab}\Big|_{[E^2]} + \hat{\hat{f}}^{ab}\Big|_{[E^4]} \quad \mathrm{Re}\, t^{ab+-} = \varphi^{ab+-}\Big|_{[E^2]} + \psi^{ab+-}\Big|_{[E^4]}$$

with leading-order $\bar{f}^{ab}$ and φ real, $\delta_\ell^{ab} = \arg[f_\ell^{ab}]$

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- Low-energy chiral suppression of high partial waves

$$\mathrm{Re}\ f_\ell(s) \sim \mathcal{O}(E^2) \qquad \mathrm{Im}\ f_\ell(s) \sim \mathcal{O}(E^6) \qquad \ell \geq 2$$

$$\mathrm{Re}\ t_\ell(s) \sim \mathcal{O}(E^4) \qquad \mathrm{Im}\ t_\ell(s) \sim \mathcal{O}(E^8) \qquad \ell \geq 2$$

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$$\mathrm{Re}\ t_\ell(s) \sim \mathcal{O}(E^4) \qquad \mathrm{Im}\ t_\ell(s) \sim \mathcal{O}(E^8) \qquad \ell \geq 2$$

- Unitarity condition + chiral counting yields, up to $\mathcal{O}(E^6)$

$$\delta_0^{+-} = \frac{\sqrt{\lambda_{+-}}}{s}[\varphi+\psi]_0^{+-+-} + \frac{\sqrt{\lambda_{00}}}{2s}\frac{\bar{f}_0^{00}}{\bar{f}_0^{+-}}\left\{\varphi^{00+-}\left[1 + \frac{\mathrm{Re}\hat{f}_0^{00}}{\bar{f}_0^{00}} - \frac{\mathrm{Re}\hat{f}_0^{+-}}{\bar{f}_0^{+-}}\right] + \psi^{00+-}\right\}$$

$$\delta_1^{+-} = \frac{\sqrt{\lambda_{+-}}}{s}[\varphi+\psi]_1^{+-+-}$$

$\Longrightarrow$ P-wave simpler: $\bar{f}_1^{00} = 0$ [Bose symmetry]

Isospin breaking for $K_{\ell 4}$ phase shifts

Form factors $K^+ \to \pi^a \pi^b \ell^+ \nu$ measured and analysed by NA48/2

$$F^{ab} = F_S^{ab}(s, s_l) + \cos\theta_\pi F_P^{ab}(s, s_l) + \ [\ell \geq 2]$$

with phase shifts that are arguments of $f_0^{+-} = F_S^{+-} = e^{i\delta_S^{+-}} |F_S^{+-}|$

and $f_0^{00} = F_S^{00} = e^{i\delta_S^{00}} |F_S^{00}|$

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$$\delta_P^{+-} = \frac{\sqrt{\lambda_{+-}}}{s}[\varphi + \psi]_1^{+-+-} + \mathcal{O}(E^6)$$

Isospin breaking between measured $\delta_S - \delta_P$ and isospin $\delta_0^0 - \delta_1^1$ from

- $\mathcal{O}(E^2)$ for S and P : LO scattering amplitudes φ
- $\mathcal{O}(E^4)$ for S and P : NLO scattering amplitudes ψ
- $\mathcal{O}(E^4)$ for S only : LO and NLO form factors $\bar{f}$ and $\hat{f}$
- All : phase space $\sqrt{\lambda}$ (trivial)

Dispersive reconstruction of amplitudes

Dispersive representation of $\pi\pi$ amplitude

In isospin limit, $\pi^a\pi^b \to \pi^c\pi^d$ scattering amplitude

$$A^{ab,cd} = A(s|t,u)\delta^{ab}\delta^{cd} + A(t|s,u)\delta^{ac}\delta^{bd} + A(u|t,s)\delta^{ad}\delta^{bc}$$

Unitarity
Analyticity
Crossing sym.

$$\Longrightarrow \quad A(s|t,u) = A_{\rm cut}(s|t,u) + A_{\rm pol}(s|t,u) + \mathcal{O}(p^8)$$

[Knecht, Moussallam, Stern, Fuchs]

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- $A_{\rm pol}$ polynomial collecting subtraction parameters

$$A_{\rm pol} = \frac{4a_0^2\pi}{3M_\pi^2}(12M_\pi^2 - 5s) + \frac{8a_0^0\pi}{3M_\pi^2}s + \mathcal{O}(E^4)$$

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- $A_{\rm cut}$ non-analytic function with cuts from $\pi\pi$ rescattering

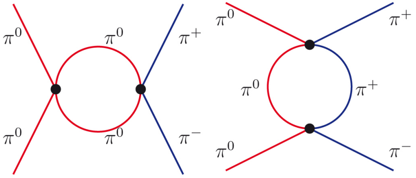
$$A_{\rm cut} = \sum_p P_p(s,t,u)\cdot W_p(s) + \sum_q Q_q(s,t,u)\cdot W_q(t) + \sum_r R_r(s,t,u)\cdot W_r(u)$$

  - P, Q, R polynomials in s, t, u and functions of subt. parameters
  - $W_i(s)$ functions with cut for $s \geq 4M_\pi^2$

$\pi\pi$ amplitude outside isospin limit

Same dispersive approach possible outside isospin limit

$$A^{+-;00}(s,t) = A_{\rm pol}^{+-;00}(s,t) - W^{(0)}_{\pm 0;\pm 0}(t) - 3(s-u)W^{(1)}_{\pm 0;\pm 0}(t) + (t \leftrightarrow u) - W^{(0)}_{+-;00}(s)$$



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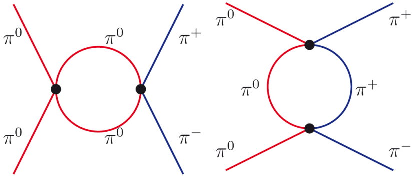
$$+(t \leftrightarrow u) - W^{(0)}_{+-;00}(s)$$



$$\frac{1}{16\pi} W^{(0)}_{\pm 0;\pm 0} = \frac{s^3}{\pi}\int \frac{dx}{x^3}\,\frac{1}{x-s}\left[\mathrm{Im}\, t_0^{+0;+0}(x) + \frac{3\left(M^2_{\pi^\pm} - M^2_{\pi^0}\right)^2}{\lambda_{\pm 0}(x)}\,\mathrm{Im}\, t_1^{+0;+0}(x)\right]$$

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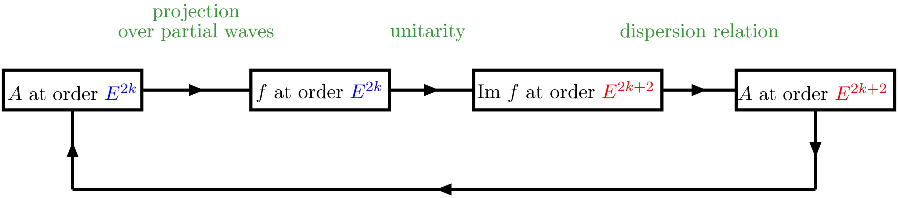
$$\frac{1}{16\pi} W^{(0)}_{+-;00} = -\frac{s^3}{\pi}\int \frac{dx}{x^3}\,\frac{\mathrm{Im}\, t_0^{+-;00}(x)}{x-s}$$

$\Longrightarrow$Other definitions of W (same cuts) amounts to redefining $A_{\rm pol}$

Iterative, dispersive reconstruction



- Yields $A = A_{\rm cut} + A_{\rm pol}$ valid up to two loops
- $A_{\rm pol}$ polynomial in s, t, u with coeff. being subtraction parameters $a, b, \lambda_i \ldots$ (2 more for each additional order and for each channel)
- $A_{\rm cut}$ collects cuts that are also functions of subtraction parameters $\Longrightarrow$Consistent, even away from a_0^0, a_0^2 predicted by χPTs
- Can be also used to reconstruct form factors
- Can be matched to χPT to constrain subtraction parameters

Starting point : partial waves

At lowest order in the chiral expansion, form factors and $\pi\pi$ amplitudes

$$F^{+-}(s) = \frac{M_K^{\pm}}{\sqrt{2}F_\pi} \qquad G^{+-}(s) = \frac{M_K^{\pm}}{\sqrt{2}F_\pi} \qquad \ldots$$

$$A^{00+-}(s,t) = 16\pi\left[a_x + b_x \frac{s-4M_{\pi^\pm}^2}{F_\pi^2}\right]$$

$$A^{+-+-}(s,t) = 16\pi\left[a_\pm + b_\pm\left(\frac{s-4M_{\pi^\pm}^2}{F_\pi^2} - \frac{t-u}{F_\pi^2}\right)\right]$$

where a's are scattering lengths, b's effective ranges

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The $\pi\pi$ partial waves are obtained by projection

$$t_0^{+-;+-} = \frac{1}{32\pi}\int_{-1}^{+1} dz\, A^{+-;+-}\left[s, t = -\frac{(s-4M_{\pi^\pm}^2)(1-z)}{2}\right]$$

$$\varphi_0^{+-;00} = 16\pi\left[a_x + b_x \frac{s-4M_{\pi^\pm}^2}{F_\pi^2}\right] \qquad \varphi_0^{+-;+-} = 16\pi\left[a_{+-} + b_{+-} \frac{s-4M_{\pi^\pm}^2}{F_\pi^2}\right]$$

First iteration : unitarity & dispersive representation

- Unitarity provides imaginary part of partial waves including $\mathcal{O}(E^4)$

$$\begin{aligned}\mathrm{Im}t_0^{+-;00}(s) &= \frac{\sqrt{\lambda_{00}}}{2s}\varphi_0^{+-;00}(s)\,\varphi_0^{00;00}(s)\,\theta(s-4M_{\pi^0}^2)\\ &+\frac{\sqrt{\lambda_{+-}}}{s}\varphi_0^{+-;+-}(s)\,\varphi_0^{+-;00}(s)\,\theta(s-4M_{\pi^\pm}^2)\,+\,\mathcal{O}(E^6)\end{aligned}$$

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- reinjected into dispersive representation of A^{+-00} with cut fncts

$$\frac{W^{(0)}_{+-;00}(s)}{(16\pi)^2} = -\frac{1}{2}\varphi_0^{+-;00}(s)\,\varphi_0^{00;00}(s)\,\bar{J}_0(s) - \varphi_0^{+-;+-}(s)\,\varphi_0^{+-;00}(s)\,\bar{J}_\pm(s) + \ldots$$

First iteration : unitarity & dispersive representation

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$$\begin{aligned}\mathrm{Im}\, t_0^{+-;00}(s) &= \frac{\sqrt{\lambda_{00}}}{2s}\varphi_0^{+-;00}(s)\,\varphi_0^{00;00}(s)\,\theta(s-4M_{\pi^0}^2)\\ &+\frac{\sqrt{\lambda_{+-}}}{s}\varphi_0^{+-;+-}(s)\,\varphi_0^{+-;00}(s)\,\theta(s-4M_{\pi^\pm}^2)\,+\,\mathcal{O}(E^6)\end{aligned}$$

- reinjected into dispersive representation of A^{+-00} with cut fncts

$$\frac{W^{(0)}_{+-;00}(s)}{(16\pi)^2} = -\frac{1}{2}\varphi_0^{+-;00}(s)\,\varphi_0^{00;00}(s)\,\bar{J}_0(s) - \varphi_0^{+-;+-}(s)\,\varphi_0^{+-;00}(s)\,\bar{J}_\pm(s) + \ldots$$

- $\bar{J}_{0,\pm}(s)$ one-loop scalar bubbles for $\pi^{0,\pm}$, with appropriate Im

$$\bar{J}_\pm(s) = \frac{s}{16\pi^2}\int_{4M_\pm^2}^\infty \frac{dx}{x}\,\frac{1}{x-s}\,\frac{\sqrt{\lambda_{+-}(x)}}{x} \to \mathrm{Im} = \frac{1}{16\pi}\,\frac{\sqrt{\lambda_{+-}(s)}}{s}\theta(s-4M_{\pi^\pm}^2)$$

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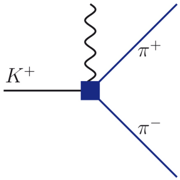
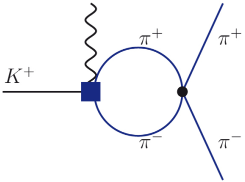
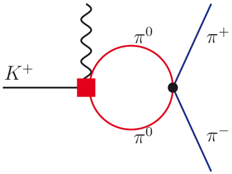
$\Longrightarrow$ Same story for $K_{\ell 4}$ form factors, involving $\pi\pi$ and πK scattering

Diagramatically : form factors







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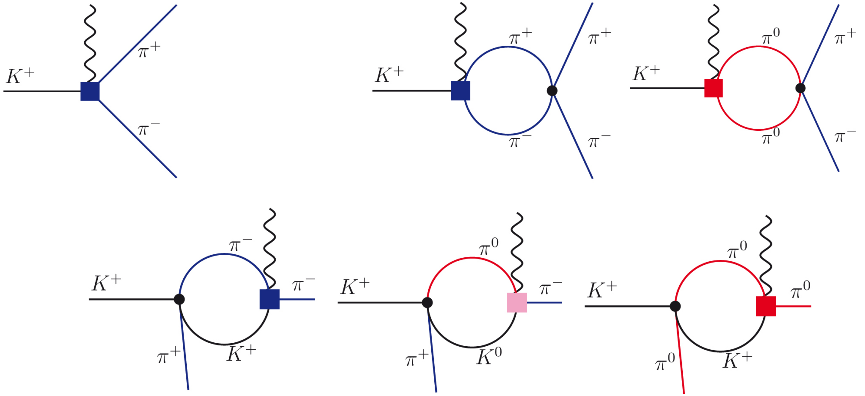

[not exhaustive]

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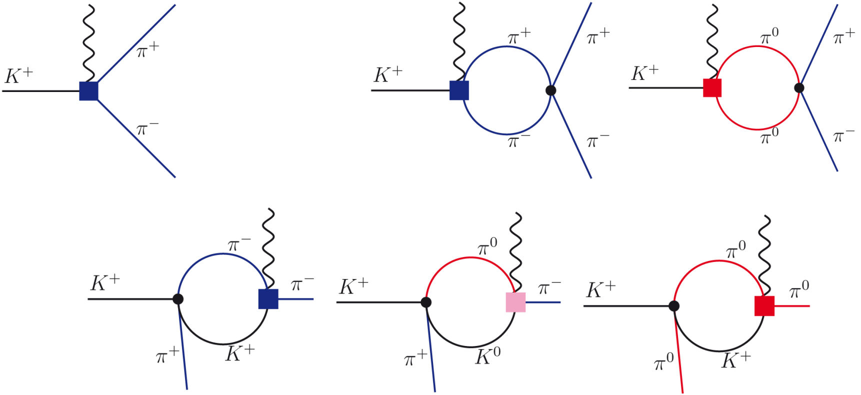
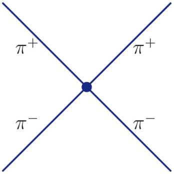


[not exhaustive]

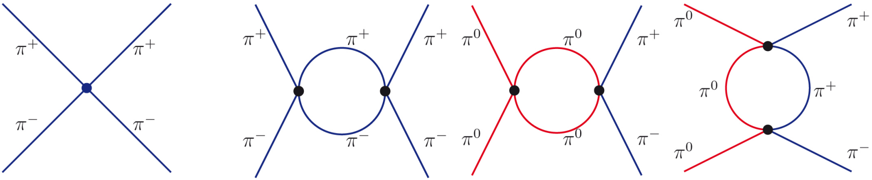
Unitarity reconstruction of form factors will involve

- s-channel: $\pi^a\pi^b \to \pi^{a'}\pi^{b'}$, $K^+ \to \pi^a\pi^b\ell^+\nu$, $(a,b) = (+,-),(0,0)$
- t, u-channels: $K\pi$ scattering, with charged or neutral kaons and the associated form factors

Diagramatically : amplitudes



Diagramatically : amplitudes



[not exhaustive]

In s, t, u-channels, unitarity reconstruction of amplitudes will involve $\pi\pi$ scattering in all channels

$$\pi^a\pi^b \to \pi^c\pi^d \text{ with}$$
$$(ab; cd) = (00; 00), (+-; 00), (+-; +-), (+0; +0), (++; ++)$$

(Half of) second iteration

- $\pi\pi$ ampl. A^X known up to 1 loop, with sub. csts $a_X, b_X, \lambda_X^{(1)}, \lambda_X^{(2)}$ for $X = (00;00), (+-;00), (+-;+-), (+0;+0), (+-;+-)$
- $K_{\ell 4}$ form factors F, G^Y known up to 1 loop, with sub. csts $\pi_{0,1,2,3}^Y$ + $\pi\pi$ and πK scattering parameters $a, b, \tilde{a}, \tilde{b}$'s for $Y = (+-), (00)$

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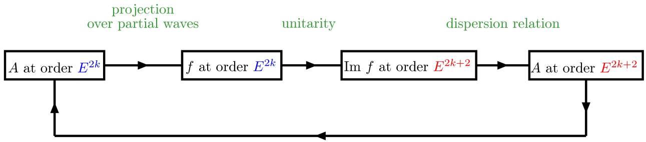

- Partial waves for NLO $\pi\pi$ ampl. and $K_{\ell 4}$ form factors [not trivial]

$$\mathrm{Re}\, t_{\ell=0,1} = \varphi_\ell + \psi_\ell + \mathcal{O}(E^6)$$

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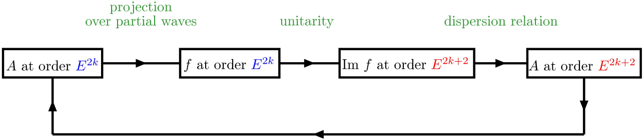

- Partial waves for NLO $\pi\pi$ ampl. and $K_{\ell 4}$ form factors [not trivial]

$$\mathrm{Re}\, t_{\ell=0,1} = \varphi_\ell + \psi_\ell + \mathcal{O}(E^6)$$

- Applying unitarity again provides enough information to reconstruct phase shifts of the form factors up to $\mathcal{O}(E^6)$

$$\mathrm{Re}\ f_{\ell=0,1} = \mathrm{Re}\ \bar{f}_\ell + \mathrm{Re}\ \hat{f}_\ell + \mathcal{O}(E^4), \quad \mathrm{Im}\ f_{\ell=0,1} = \mathrm{Im}\ \bar{f}_\ell + \mathrm{Im}\ \hat{f}_\ell + \mathrm{Im}\ \hat{\hat{f}}_\ell + \mathcal{O}(E^6)$$

Matching to one-loop χPT

Phase shifts $\delta_{S,P}(s, s_l)$ expressed in terms of loop integrals and

- $\pi\pi \to \pi\pi$ subtraction constants $a_X, b_X, \lambda_X^{(1)}, \lambda_X^{(2)}$
  for $X = (00;00), (+-;00), (+-;+-), (+0;+0), (+-;+-)$
  - matching to $\pi\pi$ one-loop amplitudes with isospin breaking
    [Knecht,Urech; Knecht,Nehme; Meißner et al.]
  - in terms of a_0^0, a_0^2 and $N_f = 2$ LECs $\bar{l}_{1,2,6}$, $\hat{k}_i$

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- $K^+ \to \pi\pi\ell\nu$ subtraction constants $\pi_{0,1,2,3}^Y$ for $Y = (+-), (00)$
  - one-loop form factors with IB [Bernard,SDG,Knecht]
  - expand massive loops ($K\bar{K}, \pi\eta \ldots$) in powers of s
  - in terms of $N_f = 3$ LECs L_i, $\hat{K}_i$ and ratio $R = \frac{m_s - \hat{m}}{m_d - m_u}$

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- $\pi K \to \pi K$ subtr. csts $\tilde{a}_Z, \tilde{b}_Z$ for $Z = (00; +-), (-+; -+), (+0, +0)$
  - πK at tree level with IB [Talavera,Nehme]
  - in terms of $a_0^{1/2}, a_0^{3/2}$ (πK), a_0^0, a_0^2 ($\pi\pi$) and ratio $R = \frac{m_s - \hat{m}}{m_d - m_u}$

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- χPT expressions with isospin breaking, but no (virtual) photons

Numerical values

- For $\bar{\ell}_i$ and L_i, standard values ($\log M_{\pi^\pm}$ for $\bar{\ell}_i$) [Gasser & Leutwyler, Bijnens et al.]
- For $\hat{k}_i$ and $\hat{K}_i$, resonance estimates even though virtual photons not included [Moussallam et al., Haefeli et al.]
- Scattering parameters $a_0^{1/2}, a_0^{3/2}$ from dispersive analysis of πK scattering [Büttiker et al.]
- Keep only first order in isospin breaking

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- Keep only first order in isospin breaking

Compute (in milliradians) the difference w.r.t isospin limit $M_{\pi^0} \to M_{\pi^\pm}$

$$\Delta = [\delta_S - \delta_0^0] - [\delta_P - \delta_1^1]$$

- LO and NLO contributions $\Delta_2\delta$ and $\Delta_4\delta$
- In NLO contribution of $\Delta\delta_S$, split
  - the part from unitarity $\Delta_4^U\delta_S$ involving only $\pi\pi$ scattering
  - the part depending on form factors $\Delta_4^F\delta_S$

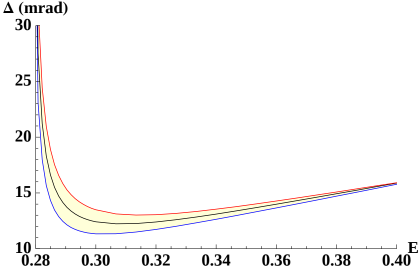
Isospin breaking for $\delta_S - \delta_P$ (Preliminary)

E (MeV)	$\Delta_2\delta_S$	$\Delta_4^U\delta_S$	$\Delta_4^F\delta_S$	$\Delta_2\delta_P$	$\Delta_4\delta_P$	Δ
286	15.75	-0.09	0.00	0.01	-0.00	15.65
296	12.91	-0.17	0.01	0.02	-0.00	12.72
305	12.52	-0.24	0.01	0.04	-0.01	12.25
313	12.60	-0.31	0.02	0.07	-0.01	12.26
322	12.88	-0.39	0.03	0.09	-0.02	12.45
331	13.27	-0.47	0.04	0.12	-0.03	12.74
340	13.75	-0.57	0.05	0.15	-0.04	13.11
351	14.36	-0.70	0.06	0.19	-0.05	13.58
365	15.17	-0.88	0.08	0.25	-0.07	14.20
390	16.80	-1.26	0.10	0.37	-0.11	15.39

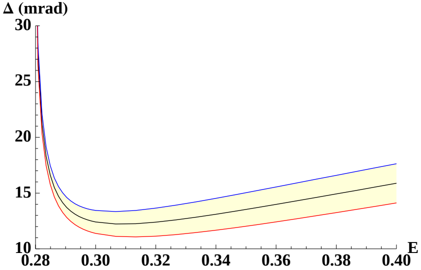
- Central values for $a_0^0 = 0.225$, $a_0^2 = -0.0382$, $R = 37$
- Slight enhancement of NLO (up to 10%) compared to LO
- Δ close to LO asymptot $\frac{7}{32\pi F_\pi^2}[M_{\pi^\pm}^2 - M_{\pi^0}^2]$ ($\sim$ 10 mrad)
- Isospin breaking in P-wave small

Dependence on a_0^0 and a_0^2 (Preliminary)

Under a shift of 20% of the isospin $\pi\pi$ scattering lengths



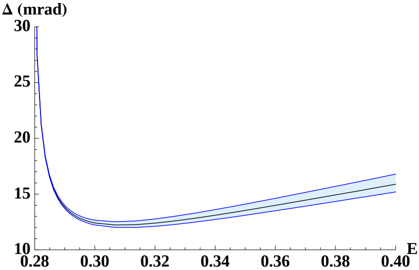
$a_0^0 = 0.225 \pm 0.045$



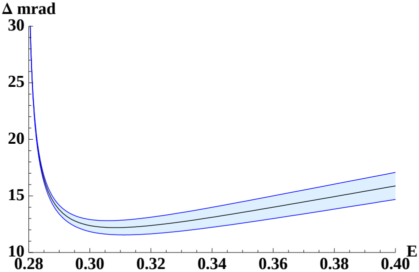
$a_0^2 = -0.0382 \pm 0.0076$

- Agreement with Gasser,Colangelo,Rusetsky for central values
- Dependence of Δ on s_l, $a_0^{1/2}$, $a_0^{3/2}$ turns out to be negligible
  (small contribution from isospin breaking in form factors)
- ... but not that on a_0^0 and a_0^2 (up to 2 mrad)

Uncertainties (Preliminary)



$R = 37 \pm 5$



$\bar{\ell}_i = \bar{\ell}_i^{cent} \pm \Delta\bar{\ell}_i, \bar{k}_i = \bar{k}_i^{cent} \pm \Delta\bar{k}_i$

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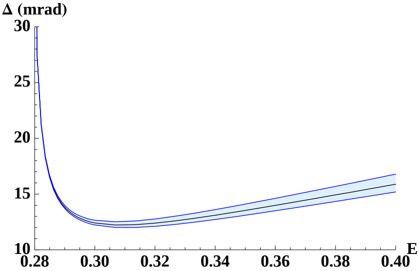

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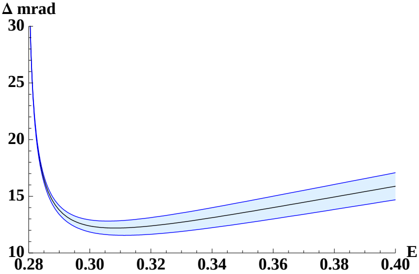

$\bar{\ell}_i = \bar{\ell}_i^{cent} \pm \Delta\bar{\ell}_i, \bar{k}_i = \bar{k}_i^{cent} \pm \Delta\bar{k}_i$

- Isospin breaking in $\delta_S - \delta_P$ parametrised with accuracy below 1 % as polynomial in

$$\left(\frac{s-4M^2_{\pi^\pm}}{M^2_{\pi^\pm}}\right)^{i-1/2} \left(\frac{a_0^0}{0.225} - 1\right)^j \left(\frac{a_0^2}{-0.0382} - 1\right)^k \left(\frac{37}{R} - 1\right)^l$$

with $i = 0\ldots 3, j = 0\ldots 2, k = 0\ldots 2, l = 0\ldots 1$

- Uncertainty from LECs as quadratic function of $(s - 4M^2_{\pi^\pm})/M^2_{\pi^\pm}$

Conclusions

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Dispersive description of $K_{\ell 4}$ decays

- Iterative approach involving $K_{\ell 4}$ form factors as well as $\pi\pi$ and πK scattering amplitudes
- description of $\pi\pi$ amplitudes and form factors up to two loops in terms of a few subtraction constants $(a, b, \ldots)$
- Matching to χPT yields subt. constants in terms of $\pi\pi$ scattering parameters a_0^0, a_0^2 in isospin limit + LECs

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Estimation of isospin breaking for phase shifts

- NLO corrections affect only marginally LO correction
- Partial cancellation between NLO corrections coming from unitarity (1 mrad) and from the form factors (0.1 mrad)
- Compatible with earlier pure χPT estimate, but ...
- ... variation in (a_0^0, a_0^2) plane can be noticeable (2 mrad)

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To be used to revisit the extraction of (a_0^0, a_0^2)
from NA48/2 $K^+ \to \pi^+\pi^-\ell^+\nu$ data