Renormalization of the three-body problem in modified Weinberg's approach at very low energies

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Outline

- Introduction;
- Three-body problem in modified Weinberg's approach;
- On the power counting for the NN potential;
- Renormalization: perturbative versus non-perturbative;
- Summary;

Introduction

At very low energies in EFT for nuclear systems one can integrate out all particles except the nucleon.

At LO of the NN interaction one is left with a constant contact interaction as the effective potential.

A non-trivial problem of renormalization arises when one uses this contact interaction in a standard non-relativistic calculation of the doublet channel *nd* scattering.

P. F. Bedaque, H. W. Hammer and U. van Kolck, Phys. Rev. Lett. **82**, 463 (1999).
P. F. Bedaque, H. W. Hammer and U. van Kolck, Nucl. Phys. A **676**, 357 (2000).

Exactly the same problem occurs in the low-energy non-relativistic EFT of self-interacting scalar particles.

Integral equation summing up an infinite number of LO diagrams does not have an unique solution.

Because of this the solution to the regularized equation does not converge to a fixed limiting value as the cutoff is removed.

Bedaque *et al.* solved the problem by introducing a three-body force at LO.

We revisited the problem in modified Weinberg's approach to EFT in E. Epelbaum, J. G., U. G. Meißner and D. L. Yao, arXiv:1611.06040 [nucl-th].

Modified Weinberg's approach suggested in
E. Epelbaum and J. G., Phys. Lett. B **716**, 338 (2012).

- Lorentz invariance is a fundamental symmetry of any EFT.
- At low energies it is useful to use expansions v/c.
- Non-relativistic series are reproduced in HB EFT
  V. Bernard, N. Kaiser and U.-G. Meißner, Int. J. Mod. Phys. E **4**, 193 (1995).
- One needs to add to the effective Lagrangian extra terms which take care of the non-commutativity of the non-relativistic expansion and loop integration.
- These additional terms lead to shifts of the coupling constants of structures already present in the effective Lagrangian.
- Non-relativistic approach exactly reproduces the expansions of the Lorentz invariant results.

- However, UV behaviour is qualitatively different in Lorentz-invariant and non-relativistic theories.
- E.g., OPE potential is perturbatively renormalizable in Lorentz-invariant formulation and non-renormalizable in HBChPT.
- While one can always add a sufficient number of counter term contributions to any finite number of iterations of the OPE potential in the non-relativistic theory, this is no longer possible when one is solving an integral equation - corresponding to an infinite number of iterations.

Three-body problem in modified Weinberg's approach

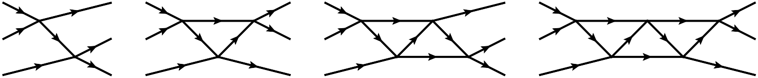
Consider a low-energy EFT of self-interacting scalar particles described by manifestly Lorenz-invariant Lagrangian

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu \phi(x) \partial^\mu \phi(x) - \frac{1}{2}\,m^2\,\phi^2(x) - \frac{\lambda}{4!}\,\phi^4(x) + \mathcal{L}_{ho}(x)\,,$$

where m and λ are the mass and the coupling of the LO interaction. $\mathcal{L}_{ho}$ contains an infinite number of Lorentz-invariant self-interactions.

At low energies it is convenient to use the TOPT.

For a large two-body scattering length, the LO amplitude of three bosons going in three bosons is given by an infinite number of diagrams, containing only particles (i.e. no antiparticles) in intermediate states, examples of which are shown below



a) *b)* *c)* *d)*

Examples of diagrams contributing to the particle-bound state scattering amplitude at leading order.

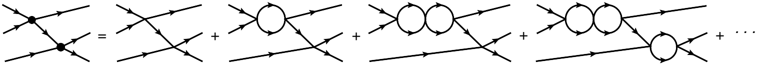
The LO effective Lagrangian describes the perturbatively renormalizable ϕ^4-theory and, therefore, none of these diagrams require counter terms beyond the LO Lagrangian.

In the non-relativistic theory, adding one more loop adds one more power to the overall degree of divergence.

Thus, the non-relativistic problem, unlike its Lorentz-invariant counter part, turns out to be perturbatively non-renormalizable.

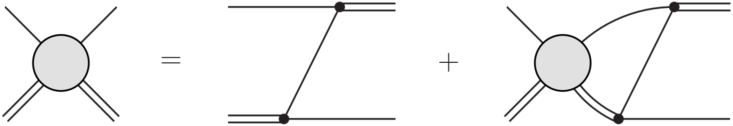
EFT renormalization of the three-body amplitude in a non-relativistic approach, which makes the non-perturbative expression cutoff independent and generates, if expanded, a perturbative series of subtracted diagrams, requires the inclusion of contributions of an infinite number of three-body counter terms.

Subsets of the infinite number of diagrams, like ones shown below, sum up to diagrams where the four particle interaction vertex is replaced by the two-body scattering amplitude.



Examples of loop diagrams which sum up to a diagram with a two-body scattering amplitude as an effective vertex. effective vertex is indicated by a filled circle.

The LO particle-bound state scattering amplitude satisfies the equation:



Solid and doubled lines correspond to the particle and the bound state, respectively.

For the amplitude in the S-wave we have:

$$
\begin{aligned}
t(k,p) &= \frac{1}{pk}\ln\frac{\omega(k)+\omega(p)+\sqrt{(k+p)^2+m^2}-E}{\omega(k)+\omega(p)+\sqrt{(k-p)^2+m^2}-E} \\
&+ \frac{2}{\pi}\int_0^\infty \frac{dq\, q^2 t(k,q)}{-\frac{1}{a_2}+J[P^2]}\frac{m}{pq\,\omega(q)} \\
&\times \ln\frac{\omega(q)+\omega(p)+\sqrt{(q+p)^2+m^2}-E}{\omega(q)+\omega(p)+\sqrt{(q-p)^2+m^2}-E},
\end{aligned}
$$

where

$$\omega(k)=\sqrt{m^2+\vec{k}^{\,2}},\ \omega_B(k)=\sqrt{(2m-B_2)^2+\vec{k}^{\,2}},$$

$$E=\omega(k)+\omega_B(k),\quad P^2=[E-\omega(q)]^2-\vec{q}^{\,2}$$

$\vec{k}$-momentum in c.o.m. frame. B_2-two-body binding energy.

$$J[P^2]=-\frac{m}{\pi P^2}\sqrt{P^2(P^2-4m^2)}\ln\left[1-\frac{P^2}{2m^2}+\frac{1}{2m^2}\sqrt{P^2(P^2-4m^2)}\right].$$

Performing $1/m$ expansion we obtain the non-relativistic equation

$$
\begin{aligned}
t(k,p) &= \frac{1}{pk}\ln\frac{k^2+p^2+kp-mE_{\rm nr}}{k^2+p^2-kp-mE_{\rm nr}} \\
&+ \frac{2}{\pi}\int_0^\infty \frac{dq\, q^2\, t(k,q)}{-\frac{1}{a_2}+\sqrt{3q^2/4-mE_{\rm nr}}}\,\frac{1}{pq}\ln\frac{q^2+p^2+qp-mE_{\rm nr}}{q^2+p^2-qp-mE_{\rm nr}},
\end{aligned}
$$

where $E_{\rm nr} = 3k^2/4m - B_2$.

It is equivalent to the S-TM equation

G. V. Skornyakov and Ter-Martirosyan, Sov. Phys. JETP **4**, 648 (1957) [Zh. Eksp. Teor. Fiz. **31**, 775 (1956)].

S-TM equation does not have an unique solution

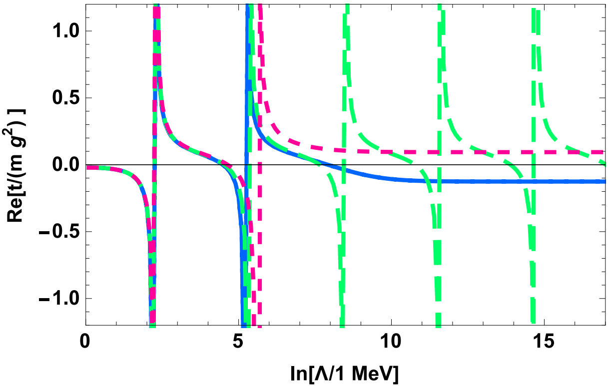
G. S. Danilov, Sov. Phys. JETP **13**, 349 (1961).

By considering a cutoff-regularized equation one obtains a unique solution, however, the removed cutoff limit does not exist.

New equation has milder ultraviolet behaviour:
For large q its integrand contains an additional factor of $\ln q$ in the denominator.

The cutoff dependence of solutions to the new and S-TM equations is shown in next slide.

The cutoff dependence of the modification of S-TM equation by including an additional factor of $1/(1+\ln(1+q/m))$ in the integrand is also shown.



Cutoff dependence of $t(k,p)$ for $a_2 = 1\,\mathrm{MeV}^{-1}$, $m = 939$ MeV, $k = 0$ MeV and $p = 10$ MeV. The solid line corresponds to new equation. The long-dashed and short-dashed lines correspond to the solution of S-TM equation and its modification by adding a factor of $1/(1+\ln(1+q/m))$ in the integrand, respectively.

Relativistic corrections induce an effective range of the two-body interaction $r_{\text{eff}} \sim 1/m$.

Its value is not related to the range of the two-body potential.

This $r_{\text{eff}} \neq 0$ guarantees a well-defined UV limit of the three-body equation.

However, in general, it cannot describe three-body observables as they depend strongly on the range of the two-body potential for the case of a large two-body scattering length.

In an EFT with contact interactions only, the range of the interaction is encoded in the contact interaction terms with derivatives.

Consistent non-perturbative inclusion of such two-body interaction terms with derivatives in the three-body sector is not feasible.

Meaningful predictions in the 3-body system with a large two-body scattering length for the case with $a_2 \gg r_2 \gtrsim m^{-1}$ require non-perturbative inclusion of the range "correction".
While for $r_2 \ll m^{-1}$ the perturbative treatment of the range corrections should be adequate.

While the three-body interaction is not required by the UV renormalization, we do not see a possibility to make an a-priori estimation of its actual impact on low-energy observables.

It might be negligible for one physical system while its effect for another system might be very large - such a case would pose a challenge to our renormalizable approach.

As the three-body interaction is perturbatively non-renormalizable already at LO, its non-perturbative inclusion does not seem to be feasible in the framework with the removed-cutoff limit.

A natural solution is provided by the inclusion of the exchange particles.

On the power counting for the NN potential

Consider LS equation:

$$T = V + V\,G T.$$

Let us write:

$$V = V_{LO} + \tilde{V}, \quad T = T_{LO} + \tilde{T},$$

where $\tilde{V}$ and $\tilde{T}$ are of higher order and T_{LO} satisfies the equation

$$T_{LO} = V_{LO} + V_{LO}\,G\,T_{LO} \;\Rightarrow T_{LO} = (1 - V_{LO}\,G)^{-1}\,V_{LO}.$$

The LO NN scattering amplitude $T_{LO} \sim \epsilon^{-1}$.
It has two different realisations:

$V_{LO} \sim \epsilon^0, \;\; G \sim \epsilon^0, \;\; 1 - V_0\,G \sim \epsilon^1$ — Weinberg

S. Weinberg, Phys. Lett. B **251**, 288 (1990).

$V_{LO} \sim \epsilon^{-1}, \;\; G \sim \epsilon^1, \;\; 1 - V_{LO}\,G \sim \epsilon^0$ — KSW

D.B.Kaplan, M.J.Savage, M.B.Wise, Phys. Lett. B **424**, 390 (1998).
The same counting is supported by the Wilsonian RG approach of M. C. Birse, J. A. McGovern, K. G. Richardson, Phys. Lett. B **464**, 169 (1999).

Wether the renormalized couplings of the effective potential satisfy the Weinberg's or KSW power counting depends on the choice of the renormalization condition!

Renormalization: perturbative versus non-perturbative

Contact interaction potential

Consider integral equations for the NN PW scattering amplitudes

$$T_{ll'}^{sj}\left(p,p',q\right) = V_{ll'}^{sj}\left(p,p'\right) + \hbar \sum_{l''} \int_0^\infty \frac{dk\,k^2}{(2\,\pi)^3} \frac{mV_{ll''}^{sj}(p,k)\;\; T_{l''l'}^{sj}(k,p',q)}{q^2 - k^2 + i\,0^+}$$

and rewrite it symbolically as

$$T = V + \hbar\, VGT\,.$$

Expansion in $\hbar$ corresponds to the standard QFT loop-expansion.

The ${}^1S_0$ NN scattering up to NLO in pionless EFT considered in

S. R. Beane, T. D. Cohen and D. R. Phillips, Nucl. Phys. A **632**, 445 (1998)

The starting NLO potential has the form

$$V_{NLO} = c + c_2 \left(p^2 + p'^2\right) .$$

This potential is perturbatively non-renormalizable.

The corresponding on-shell amplitude reads:

$$T_{NLO}(q) = \frac{c_2 \left[\hbar\, c_2 \left(I_3 q^2 - I_5\right) - 2q^2\right] - c}{\hbar\, I\left(q^2\right)\left[c_2\left(\hbar\, c_2\left(I_5 - I_3 q^2\right) + 2q^2\right) + c\right] - \left(\hbar I_3 c_2 - 1\right)^2},$$

where using the cutoff regularization loop integrals are given by

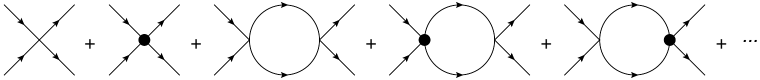
$$\begin{aligned} I_n &= -m \int \frac{d^3\vec{k}}{(2\,\pi)^3}\, k^{n-3}\,\theta(\Lambda - k) = -\frac{m\,\Lambda^n}{2\,n\,\pi^2}\,, \\ I(p^2) &= m \int \frac{d^3\vec{k}}{(2\,\pi)^3}\, \frac{1}{p^2 - k^2 + i\,0^+}\,\theta(\Lambda - k) \\ &= -\frac{i\,p\,m}{4\pi} - \frac{m\,\Lambda}{2\,\pi^2} + \frac{m\,p^2}{2\,\pi^2\Lambda} + O\left(\frac{1}{\Lambda^2}\right). \end{aligned}$$

For any finite Λ the expansion of $T_{NLO}(q)$ in $\hbar$ is a convergent series for small enough $\hbar$ and exactly coincides with the perturbative series obtained using the EFT Lagrangian which generated the potential.

In particular, the perturbative series

$$
\begin{aligned}
T_{NLO}(q) &= c + 2c_2 q^2 \\
&+ \hbar \Big[c^2 I(q^2) + c_2^2 \left(3\, I_3 q^2 + I_5 + 4\, I(q^2)\, q^4 \right) \\
&\quad + 2c_2 c \left(I_3 + 2\, I(q^2) q^2 \right) \Big] \\
&+ \hbar^2 [\cdots] \\
&+ \cdots .
\end{aligned}
$$

is in one-to-one correspondence to the diagrams



Diagrams contributing to NN scattering amplitude. Filled blob stands for the NLO, i.e. c_2, interaction.

In S. R. Beane, T. D. Cohen and D. R. Phillips, Nucl. Phys. A **632**, 445 (1998) c and c_2 have been fixed by demanding that the scattering length and the effective range are reproduced.

This leads to the amplitude

$$T_\Lambda(q) = -\frac{4i\pi a\left[4a\hbar\Lambda + \pi\left(aq^2\, r_e + 2\right)\right]}{m\left[\pi\left(a^2 q^3\, r_e + 2aq - 2i\right) + 2a\hbar\Lambda(aq(2 + iq\, r_e) - 2i)\right]}.$$

This expression is finite in $\Lambda \to \infty$ limit:

$$T(q) = -\frac{4\pi/m}{-\frac{1}{a} + \frac{q^2 r_e}{2} - iq}.$$

$T_\Lambda(q)$ is restricted by Wigner bound - cutoff cannot be taken very large unless $r_e \leq 0$ (otherwise c and c_2 become complex).

The result depends on the applied regularization scheme!

And all this has nothing to do with EFT!

The expansion of $T_\Lambda(q)$ in powers of $\hbar$ gives

$$T_\Lambda(q) = \frac{2\pi a \left(a q^2 r_e + 2\right)}{m} + \hbar \left[\frac{2 a^4 \Lambda q^4 r_e^2}{m} - \frac{i\pi a^2 q \left(a q^2 r_e + 2\right)^2}{m}\right] + \cdots ,$$

which is a convergent series for arbitrarily large but finite Λ and sufficiently small $\hbar$.

In this expression the order $\hbar$ term as well as all higher order terms contain positive powers of Λ.

Terms in this series correspond to partially renormalized diagrams - some positive powers of the cutoff are removed, others - are not.

$T_\Lambda(q)$ is **not** an EFT renormalized expression unless one defines EFT renormalization in such a way that there is a clear mismatch between the renormalization of perturbative series and a convergent sum of this series (for small $\hbar$).

J. G., Phys. Lett. B **429**, 227 (1998).

Loop diagrams are renormalized using the standard procedure.

Renormalization of the non-perturbative expression can be done by subtracting loop integrals, e.g., at $q^2 = -\mu^2$, and substituting the bare couplings c and c_2 with renormalized ones - c_R and c_{2R}.

The final result reads:

$$T_{NLO}(q) = \frac{c_R + 2q^2 c_{2R}}{1 - \hbar \left[I\left(q^2\right) - I(-\mu^2)\right]\left(c_R + 2q^2 c_{2R}\right)}.$$

Its expansion in $\hbar$ coincides to the renormalized series of diagrams.

This expression does *not* depend on the applied regularization scheme and there is *no* restriction on the sign of the effective range.

Wigner bound does not apply because the whole series of renormalized diagrams is equivalent to including the contributions of an infinite number of counter terms contained in:

$$
\begin{aligned}
V_B &= \frac{c_R - \hbar c_{2R}^2 I\left(-\mu^2\right)\left(q^2 - p^2\right)\left(q^2 - p'^{\,2}\right) + c_{2R}\left[p^2 + p'^{\,2} + \hbar c_{2R}\left(\right.\right.}{\left(\hbar I_3 c_{2R} + 1\right)^2 + \hbar I\left(-\mu^2\right)\left[c_R - c_{2R}\left(\hbar I_5 c_{2R} - q^2\left(\hbar I\right.\right.\right.} \\
&= c_R + c_{2R}(p^2 + p'^{\,2}) + \mathcal{O}(\hbar).
\end{aligned}
$$

This expression satisfies the standard Weinberg power counting applied to renormalised potential $c_R + c_{2R}(p^2 + p'^{\,2})$ (for $\mu \sim$ hard scale), while all other terms are proportional to $\hbar$.

Counter terms contain higher orders of momenta/energy, but that does not cause a problem as we are interested in power counting for physical quantities.

What is mapped to the power counting for the physical amplitude is the power counting for the potential with renormalized couplings, not the counter terms.

To summarise, the non-perturbative expression, when expanded in $\hbar$, reproduces the perturbative series and the corresponding renormalized non-perturbative expression, when expanded in $\hbar$, reproduces the renormalized perturbative series.

All this will be discussed in forthcoming paper (in an understandable and civilised form)

E. Epelbaum, A. M. Gasparyan, J. G., Ulf-G. Meißner,

"How (not) to renormalize inetgral equations with singular potentials in effective field theory".

Including OPE

According to Weinberg's counting in EFT with pions and nucleons the LO NN potential is given by (CI plus OPE)

$$V_{LO} = V_C + V_\pi,$$

where the contact interaction part V_C contributes only in S-waves.

However following the approach of
A. Nogga, R. G. E. Timmermans, and U. van Kolck, Phys. Rev. C **72**, 054006 (2005)
let us instead include a single contact interaction term in each attractive triplet PW and fix them by minimising the cutoff dependence of the PW amplitudes.

Leading order PW LS equations read

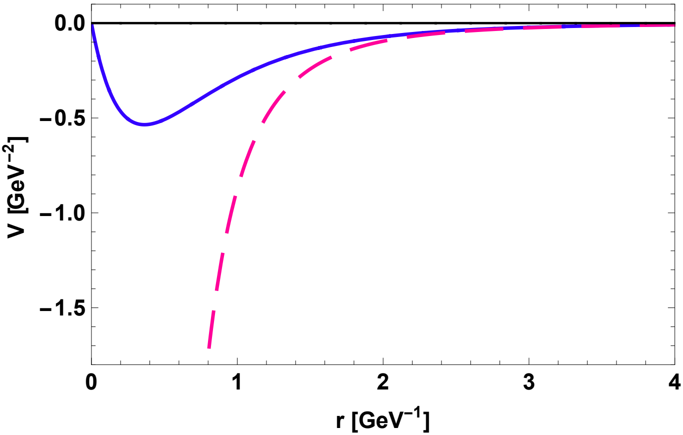
$$T_{LO} = V_{LO} + \hbar\, V_{LO} G T_{LO},$$

to which we apply the cutoff regularization.

- For large but finite cutoff Λ and small $\hbar$ iterations generate a perturbative series which converges to the solution of the equation.
- That is, for arbitrarily large but finite cutoff the solution to LS equation can be expanded in convergent series of $\hbar$.
- This series exactly reproduces the results for diagrams obtained by iterating the OPE potential and CI.
- Each term in this convergent series of diagrams can be renormalized using standard subtractive renormalization.
- We do not know how to sum up the renormalized series!
- Simple UV counting makes it clear that available contact interactions cannot generate all subtractions of loop diagrams.

- That is, for any finite cutoff the expansion of the "non-perturbatively renormalised" solution in $\hbar$ generates a convergent series which is only partially renormalised – its terms contain positive powers of the cutoff.
- These terms do not coincide to renormalized diagrams!
- One may take the cutoff to infinity in non-perturbative solution, in which case it becomes a non-analytic function of the coupling constant of the OPE potential
  W. Frank, D. J. Land and R. M. Spector, Rev. Mod. Phys. **43**, 36 (1971).
- That is, the amplitude cannot be expanded perturbatively.

- One might claim that non-perturbative expressions have nothing to do with perturbation theory because it is a solution in intrinsically non-perturbative regime.
- Notice however that this non-analyticity in the coupling constant of the OPE potential originates from the singular $1/r^3$ behaviour of OPE potential for $r \to 0$.
- The OPE potential of chiral EFT is obtained for large r and its singular $1/r^3$ behaviour for $r \to 0$ has nothing to do neither with EFT, nor QCD, which is believed to describe the real world where the only bound state of the NN system is the deuteron, while the singular attractive $1/r^3$ behaviour inevitably entails deeply bound states.



Singular and non-singular potentials. Magenta and blue lines corresponds to the singular and non-singular potentials, respectively.

Summary

- In modified Weinberg's approach the renormalization of the LO three-body scattering amplitude does not require a three-body force.
- Weinberg's power counting for NN potential is **not** inconsistent!
- In EFT non-perturbatively renormalized expressions, when expanded, reproduce renormalized perturbative series.