Recent Status of the Dual Parametrization of GPDs

Kirill Semenov-Tian-Shansky*

*IFPA, AGO, Université de Liège

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Outline

- Introduction
- Conformal PW expansion: Mellin-Barnes technique and the dual parametrization
- Basic facts on the dual parametrization
- Abel transform tomography and GPD quintessence function
- Dispersion relations and GPD sum rules.
- Analyticity in conformal spin. Story of $J = 0$ “fixed pole”
- Conclusions and outlook

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- GPD modelling can be done in various representations: (DD representation, conformal PW expansions, expansions over Bernstain polynomials,...)

List of non-trivial requirements:

- polynomiality
- hermiticity
- T-invariance
- positivity

Other sources of inspiration:

- evolution properties
- relation to PDFs and FFs
- analyticity
- Regge theory insight

- Should be possible to map one representation to another (as long as basic properties are satisfied).
- "Which representation is better is not a meaningful question!" (see K. Kumerički & D. Müller'09).
- The hope: get more insight from considering various GPD properties within different representations.

Conformal PW expansion for GPDs I

- Idea: expand GPDs over the conformal basis (factorization of functional dependencies)
- Main advantage: trivial solution of the LO evolution equations.

- Conformal moments of quark GPDs are defined with respect to $c_n(x,\xi) = N_n \times \xi^n C_n^{\frac{3}{2}}\left(\frac{x}{\xi}\right)$; Normalization: $\lim_{\xi\to 0} c_n(x,\xi) = x^n$.

$$m_n(\xi,t) = \int_{-1}^{1} dx\, c_n^{\frac{3}{2}}\left(\frac{x}{\xi}\right) H(x,\xi,t)\,.$$

- $c_n(x,\xi)$ form a complete basis in $[-\xi,\ \xi]$ with the weight $\left(1-\frac{x^2}{\xi^2}\right)$.
- $p_n(x,\xi)$ include the weight and θ to ensure the support:

$$p_n(x,\xi) = \xi^{-n-1}\theta\left(1-\frac{x^2}{\xi^2}\right)\left(1-\frac{x^2}{\xi^2}\right) N_n^{-1}\frac{(n+1)(n+2)}{2n+3} C_n^{\frac{3}{2}}\left(\frac{x}{\xi}\right)\,.$$

- Orthogonality of the basis: $$\int_{-1}^{1} dx\, p_n(x,\xi) c_n(x,\xi) = \delta_{mn}$$

Conformal PW expansion for GPDs II

Conformal PW expansion for GPDs:

$$H(x,\xi,t) = \sum_{n=0}^{\infty} p_n(x,\xi) m_n(\xi,t)\,.$$

- Allows to factorize x, ξ and t dependence of GPDs.
- Conformal moments are reproduced by this series.
- Restricted support property $\not\Rightarrow$ GPD vanishes in the outer region.
- The expansion is to be understand as an ill-defined sum of generalized functions.

Different ways to assign meaning to conformal PW expansion

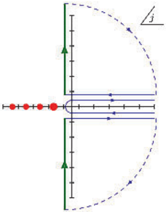
1. Sommerfeld-Watson transform + Mellin-Barnes integral techniques D. Müller and A. Schäfer'05; A. Manashov, M. Kirch and A. Schafer'05;
2. - Shuvaev transform A. Shuvaev'99, J. Noritzsch'00;
   - Dual parametrization of GPDs M. Polyakov and A. Shuvaev'02;

Mellin-Barnes techniques in simple words

- Sommerfeld-Watson transform:

$$H(x,\xi,t) = \frac{1}{2i}\oint_{(0)}^{(\infty)} dj\, \frac{(-1)^j}{\sin \pi j}\, p_j(x,\xi)\, m_j(\xi,t)\,.$$

- Residue theorem leads to conformal P.W. expansion $(\mathrm{Res}_{j=n}\frac{1}{\sin \pi j} = \frac{(-1)^j}{\pi})$.



- For $\xi = 0$ p_j form the integral kernel for the inverse Mellin transform
- In general, $p_j(x,\xi)$ are expressed through ${}_2F_1$ hypergeometric function. Asymptotic behavior of $p_j(x,\xi)$ for $j \to \infty$ is known.
- Asymptotic behavior of m_j -?
- Integral over the large arc must vanish.

- Mellin-Barnes integral representation for GPDs:

$$H(x,\xi,t) = \frac{i}{2}\int_{c-i\infty}^{c+i\infty} dj\, \frac{(-1)^j}{\sin \pi j} p_j(x,\xi) m_j(\xi,t)\,.$$

Starting point for D. Müller et al.

The basis for the Shuvaev transform & the dual parametrization

- How to restore $f(x)$ from its Mellin moments $M_n = \int dx x^n f(x)$?

- Formal solution:

$$f(x) = \sum_{n=0}^{\infty} M_n \delta^{(n)}(x) \frac{(-1)^n}{n!} .$$

✓ A trick: $\delta^{(n)}(x) = \frac{(-1)^n n!}{2\pi i} \left[\frac{1}{(x - i\epsilon)^{n+1}} - \frac{1}{(x + i\epsilon)^{n+1}} \right].$

Define $F(z) = \sum_{n=0}^{\infty} \frac{M_n}{z^{n+1}}$; then $f(x) = \frac{1}{2\pi i} \left[F(x - i\epsilon) - F(x + i\epsilon) \right]$.

Idea of the Shuvaev transform (see A. Shuvaev'99, J. Noritzsch'00):

- Introduce $f_\xi(y)$ whose Mellin moments generate Gegenbauer moments of GPD:

$$\int_0^1 dy y^n f_\xi(y) = m_n(\xi)$$

- One can explicitly construct the kernel $K(x, \xi; y)$ such that

$$H(x, \xi) = \int_0^1 dy \, K(x, \xi; y) \, f_\xi(y) \, .$$

Dual Parametrization: basic facts

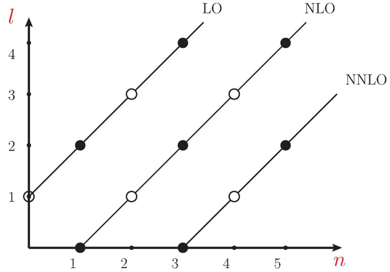
Dual Parametrization (M. Polyakov, A. Shuvaev'02):

- Mellin moments expanded in a set of suitable orthogonal polynomials. E.g. partial waves of the t-channel (t-channel refers to $\bar{h}h \to \gamma^*\gamma$):

$$N_n^{-1}\frac{(n+1)(n+2)}{2n+3}m_n(\xi,t) = \xi^{n+1}\sum_{l=0}^{n+1} B_{nl}(t)P_l\left(\frac{1}{\xi}\right)$$

Conformal PW expansion is then rewritten as:

$$H(x,\xi,t) = \sum_{\substack{n=1\\ \text{odd}}}^{\infty}\sum_{\substack{l=0\\ \text{even}}}^{n+1} B_{nl}(t)\,\theta\left(1-\frac{x^2}{\xi^2}\right)\left(1-\frac{x^2}{\xi^2}\right)C_n^{\frac{3}{2}}\left(\frac{x}{\xi}\right)P_l\left(\frac{1}{\xi}\right)$$



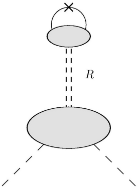
- Polynomiality implemented via Wigner-Ekkart theorem ($l \leq n+1$).
- Discrete symmetries (C, T) through the selection rules for l^{PC} (X. Ji, R. Lebed'01).
- Generalized FFs $B_{nl}(t)$ are renormalized multiplicatively.

t-channel point of view and duality

- Conformal PW expansion converges for $\xi > 1$.
- By means of the crossing relation one gets conformal PW expansion for two particle GDAs.

$$\frac{x}{\xi} \leftrightarrow 1-2z; \quad \frac{1}{\xi} \leftrightarrow 1-2\zeta; \quad t \leftrightarrow W^2$$

- Duality in the spirit of R. Dolen, D. Horn, C. Schmid'67. GPDs are presented as infinite series of t-channel Regge exchanges M. Polyakov'98:



$$\langle \pi(p')|\,\hat{O}\,|\pi(p)\rangle \sim \text{ Crossing of } \sum_{R_J} \sum_{\substack{\text{polarization} \\ \text{of } R_J}} \frac{1}{t - M_{R_J}^2}$$

$$\times \underbrace{\langle \pi(p')\pi(-p)|R_J\rangle}_{R_J\pi\pi \text{ effective vertex}} \underbrace{\langle R_J|\,\hat{O}\,|0\rangle}_{\text{F.T. of DA of } R_J} .$$

- Expansion in the t-channel PW:

$$\cos\theta_t = \frac{s-u}{\sqrt{1-\frac{4\,m^2}{t}}\,(Q^2+t)} = \frac{1}{\xi\sqrt{1-\frac{4m^2}{t}}} + O(\frac{1}{Q^2})\,,$$

- Same idea as the Shuvaev transform: Mellin moments of $Q_k(y,t)$ generate the generalized F.Fs. B_{nl}.

$$B_{n\ n+1-k}(t) = \int_0^1 dy y^n Q_k(y,t)\,.$$

Then $$H(x,\xi,t) = \sum_{k=0}^{\infty} \int_0^1 dy K^{(k)}(x,\xi,y) Q_k(y,t)\,.$$

How to construct the convolution kernels?

- M. Polyakov and A. Shuvaev'02 (see also M. Polyakov and KS'08):

$$K^{(k)}(x,\xi,y) = \mathrm{disc}_{z=x} F^{(k)}(z,\xi,y)\,, \qquad \text{where}$$

$$F^{(k)}(z,\xi,y) = \frac{1}{y}\left(1 + y\frac{\partial}{\partial y}\right)\int_{-1}^{1} ds \xi^k \frac{z_s^{1-k}}{\sqrt{z_s^2 - 2z_s + \xi^2}}\,, \quad z_s \equiv 2\frac{z-\xi s}{(1-s^2)y}.$$

Two ways to compute the discontinuity:

1. Expand in powers of $\frac{1}{z_s}$ and employ Rodriguez formula for Gegenbauer polynomials $\Rightarrow$ formally recover conformal PWE for GPD.
2. Consider the discontinuity due to the cut $1-\sqrt{1-\xi^2} < z_s < 1+\sqrt{1-\xi^2}$ (and from poles at $z_s = 0$ for $k \geq 2$) $\Rightarrow$ analytical expressions for the convolution kernels in terms of elliptic integrals.

Basic properties

- GPDs satisfy polynomiality property and the support property.
- The D-term is the natural ingredient of the dual parametrization.
- Scale dependence of $Q_k(x)$ is given by DGLAP equations.
- $Q_0(x)$ is fixed in terms of (t-dependent) PDFs:

$$Q_0(x) = q(x) + \bar{q}(x) - \frac{x}{2} \int_x^1 \frac{dy}{y^2} \left(q(y) + \bar{q}(y) \right);$$

- $Q_2(x)$ contains FFs of the EMT (J^q, shear forces)
- x-dependence of forward like functions should implement the insight from the Regge theory
- A principle allowing to take into account only a finite number of conformal PWs (i.e. Q_ks)?

Convolutions with hard kernels

- Extraction of the information on GPDs from the Compton F.Fs is the problem of deconvolution.
- Consider the elementary amplitude:

$$A(\xi,t)=\int_0^1 dx H(x,\xi,t)\left[\frac{1}{\xi-x-i0}-\frac{1}{\xi+x-i0}\right]=4\sum_{\substack{n=1\\ \text{odd}}}^{\infty}\sum_{\substack{l=0\\ \text{even}}}^{n+1}B_{n\,l}(t)P_l\left(\frac{1}{\xi}\right)$$

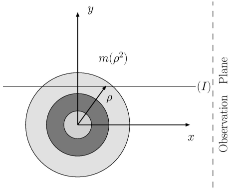
$$\mathrm{Im}A(\xi,t)=2\int_{\frac{1-\sqrt{1-\xi^2}}{\xi}}^{1}\frac{dx}{x}\,N(x,t)\,\frac{1}{\sqrt{\frac{2x}{\xi}-x^2-1}}\,.$$

- Explicit expression also exists for $\mathrm{Re}A(\xi,t)$.
- GPD quintessence:
  $N(x,t)=\sum_{\nu=0}^{\infty}x^{2\nu}Q_{2\nu}(x,t)=Q_0(x)+x^2Q_2(x)+x^4Q_4(x)+\dots$
- The amplitude automatically satisfies the dispersion relation in $\omega=\frac{1}{\xi}$ (O. Teryaev'05) with the subtraction constant given by the D-FF:

$$D(t)=\int_0^1\frac{dx}{x}\left(\frac{1}{\sqrt{1+x^2}}-1\right)+\int_0^1\frac{dx}{x}[N(x,t)-Q_0(x,t)]\frac{1}{\sqrt{1+x^2}}$$

- GPD quintessence and D-FF is the maximal amount of info one can obtain about GPDs from the amplitude.

Abel transform tomography



The observer at ∞ looking along a line parallel to the x-axis a distance y above the origin sees the projection:

$$a(y^2) = \int_{-\infty}^{\infty} dx\, m(\rho^2) = \int_{y^2}^{\infty} d\rho^2 \frac{m(\rho^2)}{\sqrt{\rho^2 - y^2}}$$

- M. Polyakov'07: with the help of Joukowski conformal map $\frac{1}{w} = \frac{1}{2}\left(x + \frac{1}{x}\right)$ it is possible to present the relation between $\mathrm{Im}A(\xi)$ and GPD quintessence $N(x)$ in the form of the Abel integral equation.
- The inverse transform for $N(x)$:

$$N(x) = \frac{1}{\pi} \frac{x(1-x^2)}{(1+x)^{\frac{3}{2}}} \int_{\frac{2x}{1+x^2}}^{1} \frac{d\xi}{\xi^{\frac{3}{2}}} \frac{1}{\sqrt{\xi - \frac{2x}{1+x^2}}} \left\{ \frac{1}{2}\mathrm{Im}A(\xi) - \xi \frac{d}{d\xi}\mathrm{Im}A(\xi) \right\}.$$

- A message for practitioners: nice way to implement analyticity constraints.

Interpretation of GPD quintessence

$$N(x,t) = \underbrace{Q_0(x,t)}_{\text{PDFs}} + x^2 \underbrace{Q_2(x,t)}_{\text{FFs of EMT tensor}} + x^4 Q_4(x,t) + \dots$$

- Only a principle possibility to separate Q_ks via logarithmic scaling violation.
- Need for the physical interpretation of GPD quintessence (otherwise the construction seems tautological)!
- Spin J expansion of the QCD string operator:

$$\bar{\Psi}(n) P \exp\left(i \int_{-n}^{n} dz^\mu A_\mu(z) \right) \Psi(-n) =$$

$$= \sum_{J=0}^{\infty}$$ []$_J$ Y_{JM}

- For massless hadrons:

$$\int_0^1 dx x^{J-1} N(x,t) = B_{J-1\,J}(t) + B_{J+1\,J}(t) + B_{J+3\,J}(t) + \dots \equiv F_J(t).$$

- GPD quintessence is anew tool to study QCD strings. Also new possibilities for studies of nucleon excitations.
- Spoiled a bit by threshold corrections for $\beta \neq 1$. Some resummation needed?

Reparametrization procedure I

- Any GPD model can be rewritten within the dual parametrization.
- Key feature: the expansion of GPD $H(x,\xi)$ in powers of ξ around the point $\xi=0$ with fixed x ($x>\xi$).

$$
\begin{aligned}
H(x,\xi) &= H^{(0)}(x) + \xi^2 H^{(2)}(x) + \xi^4 H^{(4)}(x) + \dots \\
&= Q_0(x) + \frac{\sqrt{x}}{2}\int_x^1 \frac{dy}{y^{3/2}}\, Q_0(y) + \xi^2 \Bigg[-\frac{1-x^2}{4x}\frac{\partial}{\partial x} Q_0(x) + \\
&\frac{1}{32}\int_x^1 dy\, Q_0(y) \left\{ \frac{1}{y}\left(3\sqrt{\frac{x}{y}} + 3\sqrt{\frac{y}{x}}\right) + \frac{1}{y^3}\left(3\sqrt{\frac{y}{x}} - \left(\frac{y}{x}\right)^{\frac{3}{2}}\right)\right\} \\
&+\frac{1}{4}Q_2(x) + \frac{3}{32}\int_x^1 dy\, Q_2(y)\frac{1}{y}\left(\frac{1}{2}\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} + \frac{5}{2}\left(\frac{y}{x}\right)^{\frac{3}{2}}\right)\Bigg] + O(\xi^4)
\end{aligned}
$$

- Up to the order $\xi^{2\mu}$ this expansion involves only $Q_{2\nu}(x)$ with $\nu \le \mu$
- Assume that the expansion of GPD $H(x,\xi)$ around $\xi=0$ for $x>\xi$ calculated in the framework of a certain parametrization/phenomenological model is known: $H(x,\xi) = \phi_0(x) + \phi_2(x)\xi^2 + \phi_4(x)\xi^4 + O(\xi^6)$,
  with $\phi_{2\nu}(x) = \frac{1}{(2\nu)!}\frac{\partial^{2\nu}}{\partial\xi^{2\nu}} H(x,\xi)_{\xi=0}$.
- Allows to determine $Q_{2\nu}(x)$ order by order.

- For $Q_0(x)$ the usual expression is recovered.
- The result for $Q_2(x)$ reads:

$$Q_2(x) = \frac{2(1-x^2)}{x^2}\, q(x) + \frac{(1-x^2)}{x}\, q'(x) + \int_x^1 dy \left(\frac{-15\,x}{4\,y^4} - \frac{3}{2\,y^3} + \frac{5\,x}{4\,y^2} \right) q(y)$$
$$+\, 4\,\phi_2(x) - \int_x^1 dy\, \phi_2(y) \left(\frac{15\,x}{4\,y^2} + \frac{3}{2y} + \frac{3}{4\,x} \right).$$

- The derivation of results for Q_4, Q_6, etc is straightforward.

Some lessons

- A problem reported! Assume $q(x) \sim \frac{1}{x^\alpha}$ with $\alpha \approx 1$. Then $Q_2(x) \sim \frac{1}{x^{2+\alpha}}$ and in general $Q_{2\nu}(x) \sim \frac{1}{x^{2\nu+\alpha}}$. This leads to the possible divergences of

$$B_{2\nu-1\;\;0} = \int_0^1 \frac{dx}{x} x^{2\nu} Q_{2\nu}(x)\,.$$

- Note that $B_{2\nu-1\;\;0}$ are the lowest order Mellin moments of the forward like functions $Q_{2\nu}$ with $\nu > 0$ relevant for the calculation of GPDs. In the DVCS amplitude these $B_{2\nu-1\;\;0}$ contribute only into the D form factor. This has deep consequences.

Dual parametrization v.s. Radyushkin DD Ansatz I

Reparametrization procedure allows to establish the link between the dual parametrization of GPDs and RDDA, Radyushkin'97.

GPD is obtained as a one dimensional section of a two-variable double distribution f^q:

$$H^q(x,\xi) = \int_{-1}^{1} d\beta \int_{-1+|\beta|}^{1-|\beta|} d\alpha\, \delta(x-\beta-\alpha\xi)\, f^q(\beta,\alpha) + D\text{-term}$$

RDDA: $f^q(\beta,\alpha) = \mathrm{h}(\beta,\alpha)q(\beta)$.

$$\mathrm{h}^{(b)}(\beta,\alpha) = \frac{\Gamma(2b+2)}{2^{2b+1}\Gamma^2(b+1)} \frac{[(1-|\beta|)^2-\alpha^2]^b}{(1-|\beta|)^{2b+1}}$$

- Several first forward like functions $Q_{0,2,4}$ that reexpress Radyushkin DD Ansatz in the framework of the dual parametrization were computed.
- A way to compare: assume power-like asymptotic behavior of $q(x)$ for small x: $q(x) \sim \frac{1}{x^\alpha}$ with $1<\alpha<2$ and compare $\mathrm{Im}A(\xi)$ for $\xi \sim 0$.

Dual parametrization v.s. Radyushkin DD Ansatz II

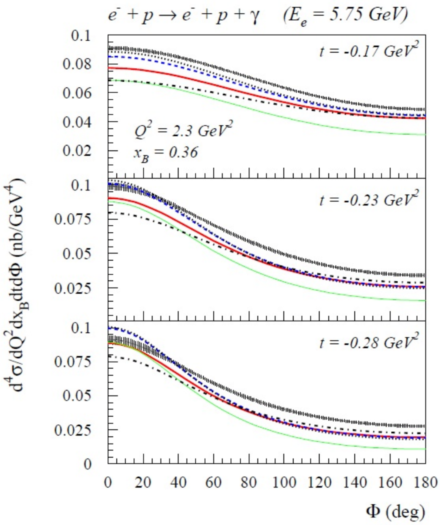
$\mathrm{Im}A(\xi)$ for $\xi \sim 0$ from $Q_{0,2,4}(x)$:

$$\mathrm{Im}A^{(0)}(\xi) + \mathrm{Im}A^{(2)}(\xi) + \mathrm{Im}A^{(4)}(\xi) + \ldots$$
$$\sim \frac{2^{\alpha+1}}{\xi^\alpha} \frac{\Gamma(\frac{1}{2})\Gamma(\alpha+\frac{3}{2})}{\Gamma(\alpha+2)} \left\{1 + (\alpha-b)\, c_2(\alpha,b) + (\alpha-b)\,(\alpha-b+1)\, c_4(\alpha,b) + \ldots\right\} .$$

$$\mathrm{Im}A_{DD}(\xi) \sim \frac{2^{2b+1-\alpha}}{\xi^\alpha} \frac{\Gamma(\frac{1}{2})\Gamma(b+\frac{3}{2})\Gamma(1+b-\alpha)}{\Gamma(2+2b-\alpha)}$$

- For $\alpha = b$ the coefficients in front of leading singular term of $\mathrm{Im}A_{DD}(\xi)$ and $\mathrm{Im}A^{(0)}(\xi)$ coincide. For small ξ the minimalist dual model is equivalent to RDDA with $b = 1$.
- For $b = \alpha + M$, $M > 0$, integer, it suffices to take account of a finite number of forward-like functions $Q_{2\nu}$ with $\nu \leq M$ obtained using the reparametrization procedure to reproduce the leading small-ξ asymptotic behavior of $\mathrm{Im}A_{DD}(\xi)$.
- The two parametrizations result in distinct behavior of $\mathrm{Im}A(\xi)$ for $\xi \sim 1$. One has to sum up all partial waves in the dual parametrization in order to reproduce $\sim (1-\xi)^b$ behavior of $\mathrm{Im}A(\xi)$ in RDDA.

Minimalist model



- The "minimalist model": $N(x,0) = Q_0(x)$ v.s. the JLab /Hall A data M. Polyakov, M. Vanderhaeghen'08

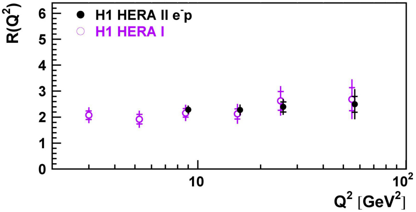
Minimalist model and skewness effect

- Consider the "minimalist model": $N(x,0) = q(x)$
- Assume that $q(x) \sim 1/x^{\alpha}$.

Skewness effect in the "minimalist" dual model equals conformal ratio (K. Kumericki, D. Mueller and K. Passek-Kumericki'08, 09)

$$r^q_{Q_0} \equiv \frac{H^q(\xi,\xi)}{H^q(\xi,0)}\Big|_{\xi\sim 0} \simeq \frac{2^{\alpha^q}\Gamma(\alpha^q+\frac{3}{2})}{\Gamma(\frac{3}{2})\Gamma(2+\alpha^q)} \approx 3/2 \quad \text{for} \quad \alpha^q \approx 1\,;$$

Skewness effect from H1:



$$\mathbf{R} = 2^{\alpha_q} r^q \sim \frac{\sqrt{\sigma_{DVCS}}}{\sigma_{DIS}}$$

The observable ratio $R(\mathrm{Q}^2)$ for fixed $W = 82$ GeV. The Figure is taken from H1'07.

Some lessons

- In order to describe the data the dual parametrization model should include some additional forward like functions $Q_{2\nu}$ with $\nu > 0$. These functions should be singular enough in order to make influence on the small ξ asymptotic behavior of $\mathrm{Im}A(\xi)$.
- Same problem in other words. Ansatz for conformal PW within Mellin-Barnes approach K. Kumericki, D. Mueller and K. Passek-Kumericki'08:

$$m_n(\xi) = \xi^{n+1} \sum_{J=J_{\min}}^{n+1} \frac{h_J}{J - \alpha(t)} P_J\left(\frac{1}{\xi}\right) .$$

In addition to the LO $\mathrm{SO}(3)$ partial wave $(J = n + 1)$ the NLO $\mathrm{SO}(3)$ partial wave should be included to fit the small-x_{Bj} experimental data.

Seems to be a problem:

- In order to contribute to the leading small-ξ singular behavior of $\mathrm{Im}A(\xi)$:

$$Q_{2\nu}(x) \sim \frac{1}{x^{2\nu+\alpha}} .$$

- This leads to divergencies of generalized form factors $B_{2\nu-1\ 0}$.
- These divergent generalized form factors contribute only into the D-form factor.

✓ Once subtracted dispersion relation in $\omega = \frac{1}{\xi}$ for the elementary amplitude reads (*e.g.* Teryaev'05):

$$A(\xi) = 4D^q + \frac{1}{\pi}\int_0^1 d\xi' \left(\frac{1}{\xi - \xi' - i\epsilon} - \frac{1}{\xi + \xi' - i\epsilon} \right) \mathrm{Im} A(\xi' - i\epsilon)\,.$$

Common wisdom:

- The subtraction constant in a dispersion relation presents an independent quantity, which cannot be fixed just with help of the information on the discontinuities of the amplitude. In order to determine the value of the subtraction constant one has to attain certain additional information on the amplitude under consideration.

A way to proceed:

- D. Mueller et al.: fix the value of the subtraction constant assume analytical properties in j of combinations of coefficients $h_{2\nu}^{(2\nu+j)}$ at powers of ξ of Mellin moments of GPD.

$$\int_0^1 dx\, x^N H_+(x,\xi) = h_0^{(N)} + h_2^{(N)}\xi^2 + \ldots + h_{N+1}^{(N)}\xi^{N+1} \quad (N = 1,\, 3,\, \ldots)\,.$$

GPD sum rule

- Dispersion relation together with the definition of the LO amplitude O. Teryaev'05, I. Anikin and O. Teryaev'07 :

$$\int_0^1 dx \left(\frac{1}{\xi - x} - \frac{1}{\xi + x} \right) [H_+(x,\xi) - H_+(x,x)] = 4D^q \,.$$

- Expansion in in powers of $\frac{1}{\xi}$ + polynomiality property $\Rightarrow$ a family of sum rules:

$$\sum_{\nu=1}^{\infty} h_{2\nu}^{(2\nu+j)} = \int_0^1 dx\, x^j \, [H_+(x,x) - H_+(x,0)] \,, \quad \text{with} \quad j = 1,\, 3,\, \ldots \,.$$

- Subtraction constant can be fixed:

$$2D^q = \sum_{\nu=1}^{\infty} h_{2\nu}^{(2\nu-1)} = \lim_{j \to -1} \left\{ \int_0^1 dx\, x^j \, [H_+(x,x) - H_+(x,0)] \right\} \,,$$

Analytical regularization

- Compute for large positive j. Then analytically continue to $j = -1$
- This is precisely a so-called analytic (or canonical) regularization ($1 < \alpha < 2$):

$$\int_{(0)}^1 dx \, \frac{f(x)}{x^{1+\alpha}} = \int_0^1 dx \, \frac{1}{x^{1+\alpha}} \, [f(x) - f(0) - x f'(0)] - \frac{f(0)}{\alpha} - \frac{f'(0)}{\alpha - 1} \,.$$

Fixing D- form factor

- Restrict the class of functions *e.g.* (I. Gelfand and G. Shilov'64):

$$z^{2\nu}Q_{2\nu}(z),\ N(z),\ \mathrm{Im}A(z) \in \left\{F:\quad F(z)=\sum_{r=1}^{R}\frac{1}{z^{\alpha_r}}f_r(z)\right\},$$

with finite R.

- The subtraction constant can be fixed according to:

$$2D^q = \int_{(0)}^{1} dx\,\frac{1}{x}\left[H_+(x,x)-H_+(x,0)\right].$$

How this applies for the dual parametrization:

$$D^q = \int_0^1 \frac{dx}{x}Q_0(x)\left(\frac{1}{\sqrt{1+x^2}}-1\right)+\int_{(0)}^{1}\frac{dx}{x}\left[N(x)-Q_0(x)\right]\frac{1}{\sqrt{1+x^2}}.$$

- This suggests the use of analytic regularization:

$$B_{2\nu-1\ \ 0} = \int_{(0)}^{1}\frac{dx}{x}x^{2\nu}Q_{2\nu}(x).$$

On the possible non analytic contributions

- The possibility to fix the D-form factor strongly relies on the postulated analyticity of Mellin moments of GPDs in Mellin space.
- Once this requirement is lifted the D-term may introduce an independent contribution into $\mathrm{Re}A(\xi)$.
- Adding of a supplementary D-term $\theta(1-\frac{x^2}{\xi^2})\,\delta D\left(\frac{x}{\xi}\right)$ with the Gegenbauer expansion:

$$\delta D(z) = (1-z^2) \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} \delta d_n \, C_n^{\frac{3}{2}}(z)$$

  to a GPD is equivalent to an introduction of the *non analytic* contributions to the forward-like functions in the framework of the dual parametrization:

$$x^{2\nu} Q_{2\nu}(x) \longrightarrow x^{2\nu} Q_{2\nu}(x) + 2\delta d_{2\nu-1}\, x\delta(x);$$

- Such situation occurs in certain dynamical models. *E.g.* pion GPD in nonlocal chiral quark model. See K.S.'08
- This results in terms "invisible" for Abel tomography like $\xi\delta(\xi)$ for $\mathrm{Im}A(\xi)$.

A tale of the $J = 0$ fixed pole

- Controversial subject since 1960s: see e.g. Creutz'73 v.s. A. Zee'72
- S. Brodsky, F. Llanes-Estrada, A. Szczepaniak'09: dispersion relation for the DVCS amplitude in $\nu = \frac{s-u}{4} = \frac{Q^2}{4\xi}$:

$$A(\xi, t) = C(t) + \xi^2 \int_0^1 \frac{dx}{x} \frac{H(x,x,t)}{\xi^2 - x^2 - i\varepsilon},$$

where the so-called "$J = 0$ pole contribution" reads

$$C(t) = \lim_{\xi \to 0} A(\xi, t) = -2 \int_{(0)}^1 \frac{dx}{x} H(x, 0, t)$$

- Key question: is $c(J) = \int_0^1 \frac{dx}{x} x^J H(x, 0, t)$ analytic in J?
  - Spin $J = 0$ exchange in the t-channel changes $c(0)$.
  - Kronecker $\delta_{J\,0}$ term also do so.
- Controversy, since in S. Brodsky, F. Llanes-Estrada, A. Szczepaniak'09 the analytic regularization is used for $C(t)$ $\Rightarrow$ no fixed poles in the terminology of the Regge theory.

Also, formally

$$4D(t) = C(t) + 2 \int_{(0)}^1 \frac{dx}{x} H(x, x, t).$$

Check of analyticity assumptions?

How can one in principle check the analyticity assumptions?

- The value of the D form factor is fixed by the small-x_{Bj} behavior of σ_{DVCS}.
- Model $N(x) - Q_0(x)$ to fit the data in the range of intermediate x_{Bj}.

Conclusions

1. The dual parametrization represents a way of handing conformal PW expansion of GPDs. To large extent it is equivalent to Shuvaev transform and Mellin-Barnes type integral based techniques.
2. Simple generalization for both quark and gluon GPDs (unpolarized, polarized and in principle helicity flip) of spin-$\frac{1}{2}$ hadrons.
3. Basic theoretical requirements hold for GPDs in the dual representation.
4. The parametrization possess several useful features useful for model builders: reparametrization procedure, Abel transform tonmography, etc. But still unable to compute α_s corrections for CFF in a closed form.
5. For small x_{Bj} the minimalist dual model is equivalent to RDDA with $b = 1$ (and leading $\mathrm{SO}(3)$ PW approximation for D. Müllers et al. approach).
6. The forward-like functions $Q_{2\nu}(x)$ with $\nu \geq 1$ may contribute to the leading singular small-x_{Bj} behavior of the imaginary part of DVCS amplitude. This makes the small-x_{Bj} behavior of $\mathrm{Im} A^{DVCS}$ independent of the asymptotic behavior of PDFs.
7. Assuming analyticity of Mellin moments of GPDs we are able to fix the value of the D-form factor in terms of the GPD quintessence function and the forward-like function $Q_0(x)$. "Duality property" of GPDs is respected.