Space-Time Structure of Polynomiality

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Deeply Virtual Compton Scattering: From Observables to GPDs

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Key Issues

- Relevance of time- and normal-ordering in operator definitions of (non-forward) parton distributions
  [previous discussions: Jaffe (1983), Diehl, Gousset (1998)]

- Relationship of the polynomiality condition for GPDs and the problem of operator ordering

- Positivity constraint and operator structure of GPDs

- Outlook

Introduction

Time-evolution:

— Heisenberg representation

$$|\Psi(t)\rangle_H = \text{const} \quad , \quad i\partial_t \hat{\phi}_H(t, \vec{x}) = [\hat{H}, \hat{\phi}_H]$$

— Interaction representation

$$i\partial_t |\Psi(t)\rangle = \hat{H}_{\rm int}(t) \, |\Psi(t)\rangle \quad , \quad i\partial_t \hat{\phi}(t, \vec{x}) = [\hat{H}_0, \hat{\phi}] \quad , \quad \hat{H} = \hat{H}_0 + \hat{H}_{\rm int} \quad ,$$

$$\hat{H}_{\rm int}(t) = \int d^3x \; \hat{\mathcal{H}}(\vec{x})$$

$$|\Psi(t)\rangle = \mathbb{U}(t, 0) \, |\Psi(0)\rangle \quad , \quad \mathbb{U}(t, 0) = \mathrm{T} \; \exp\left[-i \int_0^t dt' \hat{H}_{\rm int}(t')\right]$$

Introduction

$n-$point connected Green's function in the interaction representation

$$G^c(x_1, x_2, ...x_n) =$$

$$\langle 0|\ \mathrm{T}[\phi(x_1)\phi(x_2)...\phi(x_n)]_H\ |0\rangle = \frac{\langle 0|\ \mathrm{T}[\phi(x_1)\phi(x_2)...\phi(x_n)\ \mathbb{S}]\ |0\rangle}{\langle 0|\mathbb{S}|0\rangle}$$

$$\mathbb{S} \equiv \mathbb{U}(\infty, -\infty)$$

Heisenberg field operator $\psi_H(x)$:

$$\psi_H(x) = \mathbb{U}^\dagger(t, 0)\psi(x)\mathbb{U}(t, 0)$$

Introduction

Ordering in two-point fermion operator product

$$\mathrm{T}[\psi(x)\,\bar{\psi}(y)\,\mathbb{S}] = \Gamma^c(x,y) +$$

$$+ \sum_n \frac{(ig)^n}{n!} \int (d^4\xi)_n \sum_{\text{pairing}}{}' : \psi(x)\,\bar{\psi}(y)\,(\bar{\psi}\not{A}\psi)_{\xi_1} \ldots (\bar{\psi}\not{A}\psi)_{\xi_n} :$$

Relation between the time-ordered products of two fermion fields in the Heisenberg and in the interaction representations:

$$\mathrm{T}[\psi(x)\,\bar{\psi}(y)\,\mathbb{S}] \; = \mathbb{U}(\infty,0)\,\mathrm{T}[\psi_H(x)\,\bar{\psi}_H(y)]\,\mathbb{U}(0,-\infty)$$

Introduction

Connected fermion propagator

$$S^c(x,y) = \frac{\langle 0|\ \mathrm{T}[\psi(x)\,\bar{\psi}(y)\mathbb{S}]\ |0\rangle}{\mathbb{S}_0} = {}^H\langle 0|\ \mathrm{T}[\psi_H(x)\,\bar{\psi}_H(y)]\ |0\rangle^H$$

Hadronic matrix element of the time-ordered operator product instead of the vacuum average → the terms related to the matrix elements of the normal-ordered operators do not vanish

$$\langle p_2|\ \mathrm{T}[\psi(x)\,\bar{\psi}(y)\ \mathbb{S}]\ |p_1\rangle =$$

$$G^c(x,y)\langle p_2|p_1\rangle + \sum_{n;i,j}\int (d^4\xi)_n \langle p_2|\ :\psi(\xi_i)C_n(\xi_i,\xi_j;x,y)\bar{\psi}(\xi_j):\ |p_1\rangle +$$

$$+(\text{“}N > 2 \text{ :ordered:”})$$

Introduction

Connected matrix element of the time-ordered operator product

$$\langle p_2|\ \mathrm{T}[\psi(x)\,\bar\psi(y)\ \mathbb{S}]\ |p_1\rangle_C =$$

$$\sum_{n;i,j}\int (d^4\xi)_n \langle p_2|\ :\psi(\xi_i)C_n(\xi_i,\xi_j;x,y)\bar\psi(\xi_j):\ |p_1\rangle + (\text{“}N>2 \text{ :ordered: ”})$$

In the Heisenberg representation:

$$\sum_{n;i,j}\int (d^4\xi)_n \langle p_2|\ :\psi(\xi_i)C_n(\xi_i,\xi_j;x,y)\bar\psi(\xi_j):\ |p_1\rangle + (\text{“}N>2 \text{ :ordered:”}) \equiv$$

$$\langle p_2|\ :\psi(x)\,\bar\psi(y):\ |p_1\rangle_C^H$$

$$\langle p_2|\ \mathrm{T}[\psi(x)\bar\psi(y)\ \mathbb{S}]\ |p_1\rangle_C = \langle p_2|\ :\psi(x)\bar\psi(y):\ |p_1\rangle_C^H$$

$$\langle p_2|\ :\psi(x)\,\bar\psi(y):\ |p_1\rangle_C^H = \langle p_2|\ \mathrm{T}[\psi(x)\,\bar\psi(y)]\ |p_1\rangle_C^H$$

Factorized DVCS Amplitude

The DVCS amplitude in the interaction picture

$$\mathcal{A}_{\mu\nu} = e^2 \int d\xi d\eta e^{-iq\cdot\xi + iq'\cdot\eta} \langle p_2 | \, \mathrm{T} J^{em}_\nu(\eta) J^{em}_\mu(\xi) \mathbb{S} \, | p_1 \rangle_C$$

$$\mathcal{A} \Rightarrow \langle p_2 | : \bar{\psi}(\eta) \underline{\gamma_\nu \, S(\eta - \xi) \, \gamma_\mu} \psi(\xi) : | p_1 \rangle_C + \dots$$

$$\Phi(x, \xi) = \int d^4 k \, \delta(x - k \cdot n) \, d^4 z \; e^{i(k - \Delta/2)\cdot z} \times \langle p_2 | \, \tilde{\mathrm{T}} \bar{\psi}(0) \psi(z) \, \mathbb{S} \, | p_1 \rangle_C$$

In the Heisenberg representation:

$$\Phi(x, \xi) = \int d^4 k \, \delta(x - k \cdot n) \, d^4 z \; e^{i(k - \Delta/2)\cdot z} \, \langle p_2 | : \bar{\psi}(0) \psi(z) : | p_1 \rangle^H_C$$

$$\Phi(x, \xi) = \int d^4 k \, \delta(x - k \cdot n) \, d^4 z \; e^{i(k - \Delta/2)\cdot z} \langle p_2 | \, \mathrm{T} \bar{\psi}(0) \psi(z) \, | p_1 \rangle^H_C$$

$$= \int dk^- d^2 \mathbf{k}_T \; \Phi(xP^+, k^-, \mathbf{k}_T; \xi)$$

Factorized DVCS Amplitude

In connected matrix elements, the time-ordering and/or the normal-ordering are equivalent in the GPDs

$$\Phi^{[\gamma^+]} \stackrel{\mathrm{def}}{=} \mathrm{tr}[\gamma^+ \Phi] \Rightarrow \{H_1; H, E; ...\}$$

Consider the commutator and anti-commutator separately:

$$\Phi(x) = \Phi^{[...]}(x) + \Phi^{\{...\}}(x)$$

$$\Phi^{[...]}(x) = \frac{1}{2} \int d^4k\, \delta(x - k \cdot n)\, d^4 z e^{i(k-\Delta/2)\cdot z}\ \langle p_2|\ [\bar{\psi}(0), \psi(z)]\ |p_1\rangle_C^H$$

$$\Phi^{\{...\}}(x) = \frac{1}{2} \int d^4k\, \delta(x-k\cdot n)\, d^4 z e^{i(k-\Delta/2)\cdot z}\ \varepsilon(z_0)\, \langle p_2|\ \{\bar{\psi}(0), \psi(z)\}\ |p_1\rangle_C^H$$

Factorized DVCS Amplitude

$$\frac{1}{2}\int\limits_{-\infty}^{\infty} d^4z\, \varepsilon(z_0) e^{i(k-\Delta/2)\cdot z} \langle p_2|\ \bar{\psi}(0)\psi(z)\ |p_1\rangle_C^H$$

$$\sum_X \frac{i}{\pi}\mathcal{P}\frac{1}{k_0 - P_0 + P_0^X}\delta^{(3)}(\vec{k} - \vec{P} + \vec{P_X})\ \langle p_2|\ \bar{\psi}(0)|P_X\rangle_C^H \langle P_X|\psi(0)\ |p_1\rangle_C^H$$

If anti-commutator vanishes, the time-ordered product can be replaced by the ordinary product of operators
However: in the collinear kinematics and in the factorization regime with $t \approx 0$, the matrix element of the fermion anti-commutator does not vanish!

Factorized DVCS Amplitude:

A toy model for the box diagram

$$\gamma^*(q) + A(p_1) \to \gamma(q') + A(p_2)$$

Kinematics

$$n^2 = p^2 = 0, \quad p \cdot n = 1 \ , \quad g^T_{\mu\nu} = g_{\mu\nu} - p_\mu n_\nu - p_\nu n_\mu$$

$$p_2 = (1-\xi)p + (1+\xi)\frac{\bar{M}^2}{2} n + \Delta_T/2$$

$$p_1 = (1+\xi)p + (1-\xi)\frac{\bar{M}^2}{2} n - \Delta_T/2 \ , \quad q' = P \cdot q' n \ , \quad \bar{Q} = (q+q')/2$$

$$P = (p_1 + p_2)/2 \ , \quad \Delta = p_2 - p_1 \ , \quad P^2 = \bar{M}^2 = \frac{\Delta_T^2 - t}{4\xi^2} \ , \quad \Delta^2 = t$$

We work in the collinear kinematics: $\Delta_T \approx 0$

Factorized DVCS Amplitude:

A toy model for the box diagram

Twist-2:

$$\mathcal{A}_{\mu\nu} = \int_{-1}^{1} dx\, \mathrm{tr}[\gamma_\nu S(xP + \bar{Q})\gamma_\mu \gamma^-] \int d^4k\, \delta(x - k\cdot n)\, \Phi^{[\gamma^+]}(k) + \text{“crossed”}$$

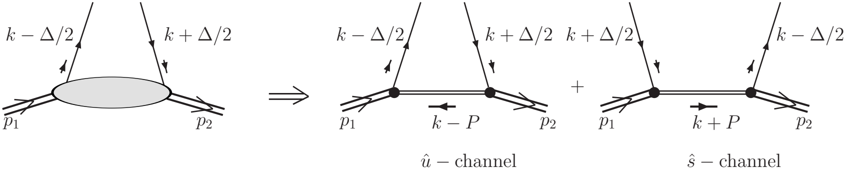
$$\Phi^{[\gamma^+]}(x, \xi) =$$

$$\int (d^4k)\, \delta(x - k \cdot n)\, \Phi^{[\gamma^+]}(k) \stackrel{g^2}{=}$$

$$ig^2 \int (d^4k)\, \delta(x - k\cdot n)\, D(k - P)[\bar{u}(p_2)\, \gamma_\alpha\, S(k + \Delta/2)\, \gamma^+\, S(k - \Delta/2)\, \gamma_\alpha\, u(p_1)]$$

Factorized DVCS Amplitude:

A toy model for the box diagram



Factorized DVCS Amplitude:
A toy model for the box diagram

Define the structure integral

$$\Phi^{[\gamma^+]}(x,\xi) = \bar{u}(p_2)\,\mathcal{I}^{[\gamma^+]}(x,\xi)\,u(p_1)\,,$$

GPD:

$$\mathcal{I}^{[\gamma^+]}(x,\xi) = \gamma^+ \int (d\mathbf{k}_T^2) \frac{\Psi^2(\mathbf{k}_T^2)}{\mathbf{k}_T^2 + \Lambda^2}\; H(x,\xi)$$

$$H(x,\xi) = \theta(-\xi < x < \xi)\left[\frac{\xi - x}{2\xi(1-\xi)} - \frac{1-x}{1-\xi^2}\right] - \theta(\xi < x < 1)\,\frac{1-x}{1-\xi^2}$$

Polynomiality of the GPDs

GPD in terms of the commutator and anti-commutator contributions

$$H^{[\cdots]}(x,\xi) = -\theta(-\xi < x < 1)\ \frac{1-x}{1-\xi^2}$$

$$H^{\{\cdots\}}(x,\xi) = \theta(-\xi < x < \xi)\ \frac{\xi - x}{2\xi(1-\xi)}$$

$$\int_{-1}^{1} dx\, x^{2n}\ H(x,\xi) = -\frac{2(1-\xi^{2n+2})}{(2n+1)(2n+2)(1-\xi^2)} = c_0 + c_2\xi^2 + \ldots + c_{2n}\xi^{2n}$$

$$\int_{-1}^{1} dx\, x^{2n+1}\ H(x,\xi) = -\frac{2(1-\xi^{2n+2})}{(2n+2)(2n+3)(1-\xi^2)} = d_0 + d_2\xi^2 + \ldots + d_{2n}\xi^{2n}$$

Polynomiality of the GPDs

Check the polynomiality for the commutator and anti-commutator contributions separately

$$\int_{-1}^{1} dx x^n \ H^{[\dots]}(x, \xi) = \frac{c_{-1}}{1-\xi} + \sum_{k=0}^{n} a_k \xi^k$$

$$\int_{-1}^{1} dx x^n \ H^{\{\dots\}}(x, \xi) = -\frac{c_{-1}}{1-\xi} + \sum_{k=0}^{n} b_k \xi^k$$

with $a_{2k-1} = -b_{2k-1}$
The anti-commutator contribution is necessary to satisfy the model independent polynomiality condition and, therefore, cannot be discarded by default

Analysis of the Positivity Constraint

Cauchy-Bunyakovsky-Schwarz inequality

$$|\langle x, y\rangle|^2 \leq \langle x, x\rangle\langle y, y\rangle$$

Formal proof

$$0 \leq \langle \lambda x + y, \lambda x + y\rangle = \lambda^2\langle x, x\rangle + 2\lambda\langle x, y\rangle + \langle y, y\rangle$$

Determinant

$$D = (2\langle x, y\rangle)^2 - 4\langle x, x\rangle\langle y, y\rangle \leq 0$$

$\rightarrow$ constraints to matrix elements

Analysis of the Positivity Constraint

Cauchy-Bunyakovsky-Schwarz inequality

$$\int d^4k\delta(x-k\cdot n)\left\{\delta((P-k)^2)\left|\lambda\,\langle P-k|\psi_+(0)|\;p_2\rangle^H+\langle P-k|\psi_+(0)|p_1\rangle^H\right|^2+\right.$$

$$\left.\delta((k+\Delta/2)^2)\left|\lambda\,\langle k+\frac{\Delta}{2},p_1|\;\psi_+^\dagger(0)\;|p_2\rangle^H+\langle k+\frac{\Delta}{2}|\;\psi_+^\dagger(0)\;|0\rangle^H\right|^2\right\}\geq 0$$

Characteristic equation

$$\lambda^2A+\lambda B+C\geq 0$$

Analysis of the Positivity Constraint

$$A = \int d^4k\delta(x - k\cdot n)\delta((P-k)^2)\ \langle p_2|\psi_+^\dagger(0)|P-k\rangle\langle P-k|\psi_+(0)|p_2\rangle^H +$$

$$\int d^4k\delta(x-k\cdot n)\delta((k+\Delta/2)^2)\langle p_2,-p_1|\ \psi_+(0)\ |k+\frac{\Delta}{2}\rangle\ \langle k+\frac{\Delta}{2}|\psi_+^\dagger(0)\ |-p_1,p_2\rangle^H$$

$$B = \int d^4k\delta(x - k\cdot n)\delta((P-k)^2)\ \langle p_2|\ \psi_+^\dagger(0)|P-k\rangle\langle P-k|\psi_+(0)|p_1\rangle^H +$$

$$\int d^4k\delta(x-k\cdot n)\delta((k+\Delta/2)^2)\ \langle p_2,-p_1|\psi_+(0)|k+\frac{\Delta}{2}\rangle\langle k+\frac{\Delta}{2}|\psi_+^\dagger(0)|0\rangle^H$$

$$+(p_1 \leftrightarrow p_2)$$

$$C = \int d^4k\delta(x - k\cdot n)\delta((P-k)^2)\ \langle p_1|\psi_+^\dagger(0)|P-k\rangle\langle P-k|\psi_+(0)|p_1\rangle^H +$$

$$\int d^4k\delta(x-k\cdot n)\delta((k+\Delta/2)^2)\ \langle 0|\psi_+(0)|k+\frac{\Delta}{2}\rangle\langle k+\frac{\Delta}{2}|\psi_+^\dagger(0)|0\rangle^H$$

Analysis of the Positivity Constraint

$$D = B^2 - 4AC \leq 0$$

$$\left[H^{[\dots]}_{S(A)}(x,\xi) + H^{\{\dots\}}_{S(A)}(x,\xi)\right]^2 \leq \left[q(x_2) + D(x_2)\right]\left[q(x_1) + C(x_1)\right]$$

$$D(x) = \int d^4k\, \delta(x - k\cdot n)\, d^4z\ e^{i(k-\Delta/2)\cdot z}\ \langle p_2, p_1|\ \psi_+(z)\psi_+^\dagger(0)\ |p_2, p_1\rangle^H$$

$$C(x) = \int d^4k\, \delta(x - k\cdot n)\, d^4z\ e^{i(k-\Delta/2)\cdot z}\ \langle 0|\ \psi_+(z)\psi_+^\dagger(0)\ |0\rangle^H$$

Conclusions

- In the collinear kinematics and in the factorization regime $t \approx 0$ the matrix element of the fermion anti-commutator is not equal to zero
- Non-vanishing of this contribution is related to the fulfilment of the polynomiality condition for the GPDs
- A new (positivity) constraint for the GPDs suggests the contributions from the forward distribution and the quark condensate